SVERIGES GEOLOGISKA UNDERSÖKNING

SER. C. Avhandlingar och uppsatser. N:o 508.

ÅRSBOK 43 (1949) N:o 6.

INTERPRETATION OF MAGNETIC ANOMALIES AT SHEET-LIKE BODIES

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Pris 8 kronor

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CONTENTS.

Introduction.

The magnetic anomaly existing at a sheet is of great importance to the practical interpretation technique. This is above all due to the fact that in nature there frequently occur well-defined magnetic bodies whose configuration nearly approaches that of a sheet. There are good examples of this among the frequently recurring dykes of basic rocks as well as among stratified iron ores. In addition, the calculation of the magnetic field at a sheet is comparatively simple.

Two different problems occur in interpretation work, viz. 1) that of determining the anomaly field when the disturbing body is known and 2) that of calculating data for the disturbing body when the anomaly is known. In the case where the disturbing body from a magnetical point of view may be considered as consisting of a single homogeneously magnetized sheet of infinite length, formulae for calculating the vertical and horizontal intensities, according to case 1) above, have been advanced by several authors (1—22). On the other hand it appears that the case of a sheet of finite length has not yet been thoroughly dealt with. The formulae dealing with simple sheets may also be applied to calculate the disturbing fields at bodies that may be regarded as composed of a limited number of homogeneously magnetized sheets.

Two principally different methods are used for the calculations involved in problem 2) above. The first may be characterized as an indirect means of approach, in which a solution is successively sought by the trial and error method. Various data of the disturbing body are assumed and from these data, anomaly values are calculated. The data are varied by increments in order to obtain progressively better agreement with the measured anomaly. This searching method may be largely facilitated by compiling suitable standard tables, and by making use of an atlas showing the appearance of the anomaly for a large number of illustrative variations in the disturbing body. The other method is a direct method of approach. Definite solutions to the values of the disturbing body parameters are found by specific calculations or geometric constructions. These calculations are based on the anomaly values at certain chosen points of an observed magnetic profile.

From the point of view of interpretation, a distinction is usually made between »thin» and »thick» magnetic sheets. The treatment of the former is much simpler than that of the latter. In practice the case of a thin sheet is applicable when its thickness is less than half the distance between the datum plane and the upper edge of the sheet.

The interpretation methods of direct calculation hitherto used, are based on special characteristic points of the vertical (Z) or horizontal (X) intensity-curves along a horizontal profile line, running at right angles to the strike of the sheet.

According to a well-known rule given by Rössiger and Puzischa (19), the depth of the upper edge of a 'thin' sheet is equal to half the distance between the two points on the profile line, where the vertical intensity is half of the maximum value. This rule is not an exact one, but the error limits are rather narrow as long as the magnetization parallel to the plane of the sheet (edge magnetization) is large compared to that at right angles to the plane (cross-magnetization). Later on Duhoux (4) indicated a simple way of estimating the position and magnetization of the sheet from a profile curve for Z, using the maximum and minimum points of the curve. Koulomzine and Massé (16) have advanced a direct method of approach for a thick sheet that is based on the maximum, minimum and inflection points of the Z-profile. Further, Hedström and Törnquist (21) have derived formulae for the determination of the magnetic parameters of a sheet of infinite length and depth by using the maximum and minimum points on a profile curve of X or Z.

Unfortunately it is not always possible to apply these methods to actual cases. The measured magnetic profile is very often more or less distorted by disturbances from magnetic sections in the body of the rock; sections that do not belong to the actual sheet. The result of this is that the positions of the extreme and inflexion points on a measured profile curve of X or Z, may deviate considerably from those which are valid for undisturbed profile curves, arising only from the sheet itself. In addition, the observation points along the profile line in terrain may be situated too far apart from each other to allow one to fix the required points on the magnetic profile curves with the necessary accuracy.

In this paper new methods are described for calculating the parameters of a thin sheet of infinite length and depth direct from the anomaly values. In these methods profile curves, representing the intensity of one or two components of the magnetic anomaly, form the basis of the calculations. The points on such profile curves necessary for the calculations may be chosen at will. The fact that sometimes a magnetic profile can be regarded as "undisturbed" within certain resstricted areas only, does not prevent these methods of calculation from being used. In addition, the methods are extended to cope with the case of a sheet having limited length, and with the case where the disturbing body may be regarded as consisting of two thin sheets or one thick sheet. The calculation technique is illustrated by actual examples. In the preamble, formulae for the anomalies at a thin sheet of infinite and finite length are derived. Numerical tables of these formulae are appended to the paper.

The interpretation studies now published have been carried out as a link in the research work pursued by the Committee for "Magnetometric studies of Iron Ores", of the Mining Research Board of Jernkontoret. The author wishes to express his sincere appreciation of the valuable aid given him in his work, and also of the permission to publish the results.

CHAPTER I

Magnetic Field at a Sheet.

§ 1. Definitions and symbols.

It is assumed in this paper, as long as nothing else is especially stated, that the sheet has a rectangular form, and may be considered as "thin" and homogeneously magnetized. The sheet is assumed to be situated in a right-handed, right-angled coordinate system xyz with the xy-plane horizontal, and the positive z-axis directed downwards.

Some terms used in the following need to be defined:

Datum point is a point where some data of the anomaly field are known.

Calculation point is a datum point used in the interpretation calculations.

Datum line is a straight line going through at least two datum points.

Reference line is a datum line used in the interpretation calculation.

Upper edge or only *edge* is that edge of the sheet which is situated nearest to the reference line.

Lower edge is that edge of the sheet which is parallel with the upper edge.

Orthogonal profile line is a straight line running at right angles to the upper edge.

Strike direction or *strike of the sheet* is the direction of the section line between the sheet plane and the xy-plane.

Head point is that point at a reference or profile line, where the vertical plane through the sheet edge cuts this line.

Natural remanence is the remanent (permanent) magnetization that would exist in a body, if the demagnetizing forces set up by the configuration of the body were zero. This quantity is a vector, i. e. fixed in magnitude and direction.

Apparent field of magnetization is the field vector that would induce the same magnetism in a body as the one existing, which is caused by the earth's magnetic field and the natural remanence.

In the following, positive and negative directions of datum, reference or profile lines are dealt with. By positive direction we mean the direction an arrow has along such a line, when its projection on the x-axis is directed towards increasing x.

When speaking of positive direction of the edge, or the strike of the sheet, or of a normal to a vertical plane through a reference line, we mean the direction an arrow has along such a line, when its projection on the y-axis is directed towards increasing y.

Several of the angles defined in the following are described as angles taken *from* one line or direction *to* another line or direction. It is thereby assumed that the angles are taken in *clockwise* direction and, if nothing else is stated, between positive directions. By clockwise direction we mean a right-handed rotation in relation to the positive direction of the rotation axis. The clockwise direction is thus counted in relation to the positive z-axis for angles in the xy-plane, in relation to the positive direction of the sheet edge for angles in the plane perpendicular to this line, in relation to the positive direction of the normal to the vertical plane through the reference line for angles in this plane and so on.

Symbols of vectors are marked out with an arrow (→).

The following symbols are used in this paper:

Dimensions and position of the sheet

ε = thickness.

$2l$ = length of the upper edge.

d = distance between the upper and lower edges.

x_0 = x-coordinate of the head point at a reference line.

z_0 = z- » » » » » » » »

$t_0 = - z_0$.

α = strike angle of the edge = angle taken from the y-axis to the projection of the edge on the xy-plane.

β = slope angle of the reference line = angle in the vertical plane through the reference line taken from a horizontal profile line to the reference line.

γ = slope angle of the edge = angle in the vertical plane through the sheet edge taken from a horizontal line to the edge.

φ = dip angle of the sheet = angle in a plane perpendicular to the strike of the sheet taken from the *negative* direction of the horizontal cutting line with the xy-plane to the direction towards the lower edge of the sloping cutting line with the sheet plane.

φ' = angle in a plane perpendicular to the sheet edge taken from the *negative* direction of the horizontal cutting line with the xy-plane to the direction towards the lower edge of the sloping cutting line with the sheet plane.

Sheet magnetization and magnetizing field

$M_{//}$ = edge magnetization = magnetization perpendicular to the edge and parallel to the plane of the sheet. $M_{//}$ is reckoned positive when the magnetization is directed from the upper edge to the lower edge, and negative when the magnetization has the opposite direction.

$M_{\perp}$ = cross magnetization = magnetization perpendicular to the plane of the sheet. $M_{\perp}$ is reckoned positive when the direction of the cross magnetization is that of increasing φ', and negative in the opposite direction.

$M_T = \sqrt{M_{//}^2 + M_{\perp}^2}$.

$\varkappa$ = magnetic susceptibility of the sheet material.

$\varkappa'_{//}$ = apparent » in the direction of $M_{//}$.

$\varkappa'_{\perp}$ = » » » » » » $M_{\perp}$.

$N_{//}$ = demagnetizing factor in the direction of $M_{//}$.
$N_{\perp}$ = » » » » » » $M_{\perp}$.
$N_{=}$ = » » » » » » the edge.
T = intensity of the earth's total magnetic field.
ϑ = declination angle of $\vec{T}$ = angle taken from the x-axis to the horizontal component $(\vec{H})$ of $\vec{T}$.
i = inclination angle of $\vec{T}$.
v_i = the angle between $\vec{T}$ and the positive direction of the edge.
$\vec{T'}$ = the projection of $\vec{T}$ on a plane perpendicular to the edge.
i' = angle in a plane perpendicular to the edge taken from the *negative* direction of a horizontal orthogonal profile line to T'.
I_r = intensity of natural remanence.
ϑ_r = declination angle of $\vec{I_r}$ = angle taken from the x-axis to the horizontal component of $\vec{I_r}$.
ψ = inclination angle of $\vec{I_r}$ = the angle between $\vec{I_r}$ and the horizontal component of $\vec{I_r}$. ψ is reckoned positive when the projection of $\vec{I_r}$ on the z-axis is directed towards increasing z, and negative in the opposite case.
v_r = the angle between $\vec{I_r}$ and the positive direction of the edge.
$\vec{I_r'}$ = the projection of $\vec{I_r}$ on a plane perpendicular to the edge.
ψ' = angle in a plane perpendicular to the edge taken from the *negative* direction of a horizontal orthogonal profile line to $\vec{I_r'}$.
T'' = intensity of the apparent field of magnetization in a plane perpendicular to the edge.
i'' = angle in a plane perpendicular to the edge taken from the *negative* direction of a horizontal orthogonal profile line to $\vec{T''}$.

$$k = \frac{I_r}{\varkappa T}, \quad k' = \frac{I_r'}{\varkappa T'}.$$

Anomaly field

$P(x, y, z)$ = point outside sheet.
$Q(a, b, c)$ = point belonging to sheet.
r = distance between points P and Q.
W = potential of total field in point $P(x, y, z)$
$W_{//}$ = part of W caused by the edge magnetization.
$W_{\perp}$ = » » » » » » cross »
X, Y, Z = intensities of the field components along the x-, y-, z-axes. Their signs follow those of the axes.
F, G, K = intensities of field components in directions determined by the direction cosines: v_x, v_y, v_z, respectively u_x, u_y, u_z, respectively s_x, s_y, s_z.
$x_i, y_i, z_i, X_i, Y_i, Z_i, F_i, G_i, K_i$ = data of the calculation point (i).

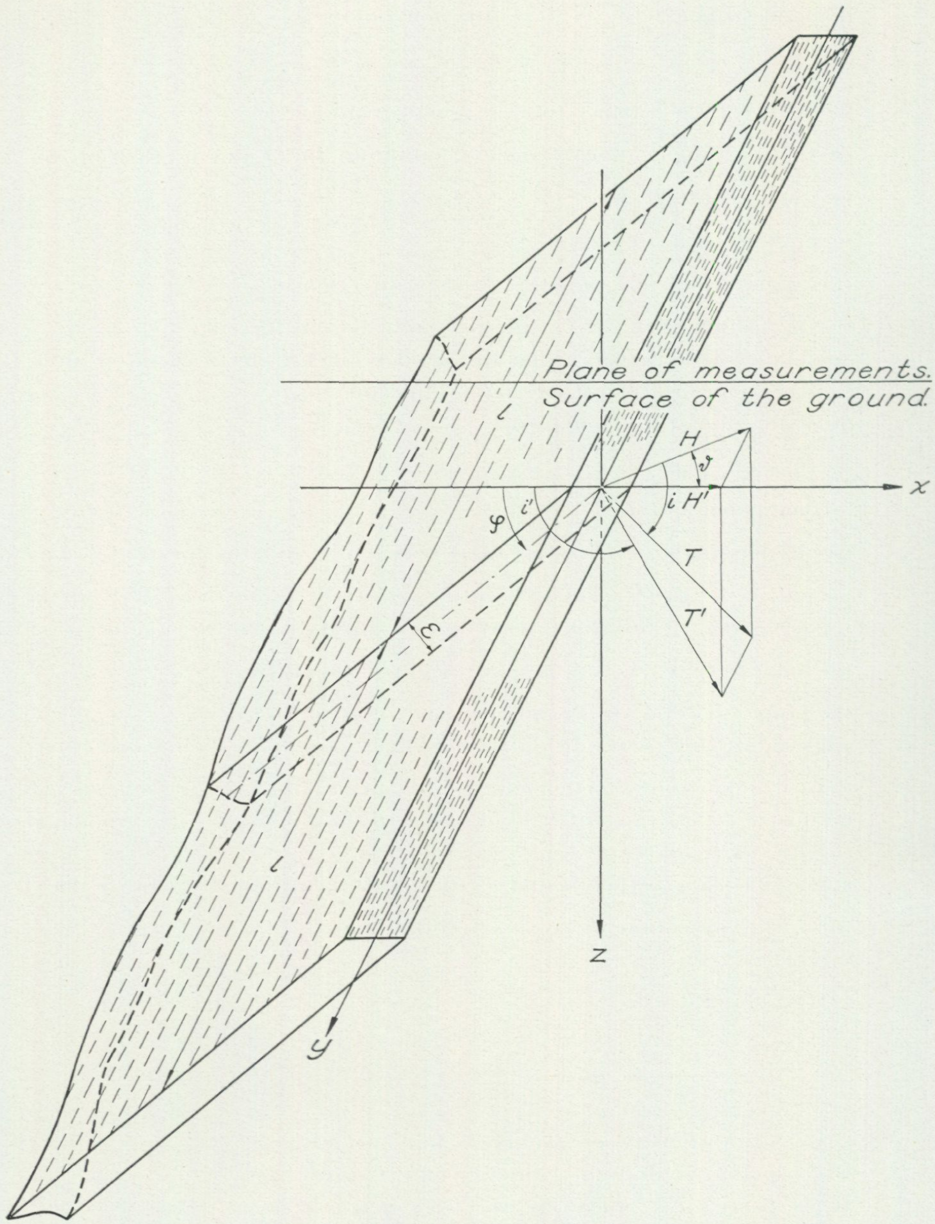

Fig. 2.1. Position of the sheet and the elements of the earth's normal field.

§ 2. Fundamental formulae.

In the following, formulae of the magnetic field at a thin, homogeneously magnetized rectangular sheet are derived. Such a sheet of finite depth may be considered as replaced by two sheets in the plane of the original one, but of infinite extent in depth, viz. one sheet with the same upper edge and the same magnetization as the original one, and the other one with its edge in line with the lower edge of the original sheet, and having a magnetization of the same magnitude but in the opposite sense. Thus it is necessary to give only the formulae of the case where the sheet extends infinitely in the dip direction.

The sheet is assumed to be situated in a coordinate system xyz according to fig. 2. 1, where the edge coincides with the y-axis. The dip angle (φ) of the sheet is taken from the negative x-axis towards the positive z-axis according to the stipulations in § 1.

The magnetization of the sheet is equivalent to an application of free magnetism to its bounding surfaces. In the following only those formulae for X and Z are derived that apply to the plane of symmetry, i. e. $y = 0$. The free magnetism at the sloping side edges of the sheet does not influence X and Z in the plane mentioned and may thus be neglected. The free magnetism on the upper edge of the sheet is $-\varepsilon M_{//}$ per unit length. At the face from which a positive cross magnetization is directed into the sheet, the amount of free magnetism is $-M_{\perp}$ per unit area; at the other face it is $M_{\perp}$.

The distance between two points $P(x, y, z)$ and $Q(a, b, c)$ is determined by the relation

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

First we treat the case when the sheet is situated vertically downwards, i. e. $\varphi = 90^\circ$.

The potential at a point $P(x, 0, z)$, caused by the free magnetism of a length element db of the edge at the point $y = b$ of the y-axis, is $dW_{//} = -\frac{\varepsilon M_{//} db}{r}$ and $a = c = 0$ for all length elements of the edge.

Thus we have $W_{//} = -\int\limits_{b=-l}^{b=l} \frac{\varepsilon M_{//}}{r}\, db$ and

$$-\frac{\partial W_{//}}{\partial x} = \varepsilon M_{//}\int\limits_{b=-l}^{b=l} \frac{\partial}{\partial x}\left(\frac{1}{r}\right) db = -\varepsilon M_{//}\int\limits_{b=-l}^{b=l} \frac{x}{r^3}\, db = -\varepsilon M_{//}\, x \left|\frac{b}{(x^2+z^2)\sqrt{x^2+b^2+z^2}}\right|_{b=-l}^{b=l}$$

$$= -2\,\varepsilon M_{//} \frac{l}{\sqrt{x^2+l^2+z^2}} \cdot \frac{x}{x^2+z^2} \quad \ldots\ldots\ldots\ldots \quad (2.1)$$

Similarly we obtain

$$-\frac{\partial W_{//}}{\partial z} = -2\,\varepsilon M_{//} \frac{l}{\sqrt{x^2+l^2+z^2}} \cdot \frac{z}{x^2+z^2} \quad \ldots\ldots\ldots\ldots \quad (2.2)$$

An element of volume $\varepsilon\, db\, dc$ at the point $Q(0, b, c)$ has a magnetic moment $\varepsilon M_{\perp}\, db\, dc$ in the positive direction of the x-axis. The potential ($dW_{\perp}$) of this element at $P(x, 0, z)$ is then according to Poisson's law

$dW_{\perp} = -\frac{\partial}{\partial x}\left(\frac{\varepsilon M_{\perp} db\, dc}{r}\right)$ and $a = 0$ for all elements of volume of the sheet.

Thus we may write $W_{\perp} = -\varepsilon M_{\perp} \int\limits_{b=-l}^{b=l} \int\limits_{c=0}^{c=\infty} \frac{\partial}{\partial x}\left(\frac{1}{r}\right) db\ dc.$

Since $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial a}\left(\frac{1}{r}\right)$ and $\left(\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial c^2}\right)\left(\frac{1}{r}\right) = 0$

we have

$$-\frac{\partial W_{\perp}}{\partial x} = -\varepsilon M_{\perp} \int\limits_{b=-l}^{b=l} \int\limits_{c=0}^{c=\infty} \left(\frac{\partial^2}{\partial b^2} + \frac{\partial^2}{\partial c^2}\right)\left(\frac{1}{r}\right) db dc$$

$$= -\varepsilon M_{\perp} \left[\int\limits_{c=0}^{c=\infty} \left| \frac{\partial}{\partial b}\left(\frac{1}{r}\right) \right|_{b=-l}^{b=l} dc + \int\limits_{b=-l}^{b=l} \left| \frac{\partial}{\partial c}\left(\frac{1}{r}\right) \right|_{c=0}^{c=\infty} db \right]$$

$$= 2\, l \varepsilon M_{\perp} \left[\frac{1}{x^2 + l^2} + \frac{z}{(x^2 + l^2)\sqrt{x^2 + l^2 + z^2}} + \frac{z}{(x^2 + z^2)\sqrt{x^2 + l^2 + z^2}} \right]$$

$$= 2\ \varepsilon M_{\perp} \frac{l}{\sqrt{x^2 + l^2 + z^2}} \left[\frac{z + \sqrt{x^2 + l^2 + z^2}}{x^2 + l^2} + \frac{z}{x^2 + z^2} \right] \quad \ldots\ldots \quad (2.3)$$

Further we obtain

$$-\frac{\partial W_{\perp}}{\partial z} = \varepsilon M_{\perp} \int\limits_{b=-l}^{b=l} \int\limits_{c=0}^{c=\infty} \frac{\partial^2}{\partial c\, \partial a}\left(\frac{1}{r}\right) db dc = \varepsilon M_{\perp} \int\limits_{b=-l}^{b=l} \left| \frac{\partial}{\partial a}\left(\frac{1}{r}\right) \right|_{c=0}^{c=\infty} db$$

$$= \varepsilon M_{\perp} \int\limits_{b=-l}^{b=l} \frac{-x}{(x^2 + b^2 + z^2)^{3/2}} db = -2 \varepsilon M_{\perp} \frac{l}{\sqrt{x^2 + l^2 + z^2}} \cdot \frac{x}{x^2 + z^2} \quad \ldots \quad (2.4)$$

Now $X = -\frac{\partial W}{\partial x} = -\frac{\partial W_{//}}{\partial x} - \frac{\partial W_{\perp}}{\partial x}$ and $Z = -\frac{\partial W}{\partial z} = -\frac{\partial W_{//}}{\partial z} - \frac{\partial W_{\perp}}{\partial z}$.

Inserting the expressions (*2.1—2.4*) for the derivatives in the right membra we obtain

$$\left.\begin{aligned} X &= -\frac{2\ \varepsilon l}{\sqrt{x^2 + l^2 + z^2}} \left[\frac{x M_{//} - z M_{\perp}}{x^2 + z^2} - \frac{M_{\perp}}{\sqrt{x^2 + l^2 + z^2} - z} \right], \\ Z &= -\frac{2\ \varepsilon l}{\sqrt{x^2 + l^2 + z^2}} \cdot \frac{z M_{//} + x M_{\perp}}{x^2 + z^2}. \end{aligned}\right\} \quad \ldots\ldots \quad (2.5)$$

From equations (*2.5*) it is simple to derive the formulae for X and Z for some arbitrary dip angle φ of the sheet. At point $P(x, 0, z)$ the components of the magnetic field in the xz-plane, parallel and perpendicular to the plane of the sheet, are designated Z' and X' respectively. Then we have

$$\left.\begin{aligned} X &= X' \sin \varphi - Z' \cos \varphi, \\ Z &= X' \cos \varphi + Z' \sin \varphi. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.6)$$

X' and Z' may be obtained from equations (2.5) by substituting x' for x and z' for z and in doing so putting

$$\left.\begin{aligned} x' &= x \sin \varphi + z \cos \varphi, \\ z' &= -x \cos \varphi + z \sin \varphi. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.7)$$

If the calculated values of X' and Z' thus obtained are inserted in equation (2.6) and if z is replaced by $-t$ we obtain

$$\left.\begin{aligned} X &= -2\varepsilon \frac{l}{\sqrt{x^2+l^2+t^2}} \left[\frac{tM_\perp + xM_{/\!/}}{x^2+t^2} - \frac{M_\perp \sin\varphi}{t\sin\varphi + x\cos\varphi + \sqrt{x^2+l^2+t^2}}\right], \\ Z &= 2\varepsilon \frac{l}{\sqrt{x^2+l^2+t^2}} \left[\frac{tM_{/\!/} - xM_\perp}{x^2+t^2} + \frac{M_\perp \cos\varphi}{t\sin\varphi + x\cos\varphi + \sqrt{x^2+l^2+t^2}}\right]. \end{aligned}\right\} \quad (2.8)$$

When $l \to \infty$, i. e. when the sheet is of infinite length, the above expressions reduce to the formulae advanced by Rössiger och Puzischa,

$$X = -2\varepsilon \frac{xM_{/\!/} + tM_\perp}{x^2+t^2}, \quad Z = 2\varepsilon \frac{tM_{/\!/} - xM_\perp}{x^2+t^2}. \quad \ldots\ldots\ldots\ldots \quad (2.9)$$

Equations (2.8—2.9) are regarded as the fundamental formulae for the field at a thin sheet.

In practice, when a calculation of the field at a given sheet is required, only parameters of dimension and position are known as a rule, and not $M_{/\!/}$ or $M_\perp$. These latter quantities may, however, be calculated if the magnetic susceptibility and natural remanence of the sheet material are known.

§ 3. Calculation of $\mathbf{M}_{/\!/}$ and $\mathbf{M}_\perp$.

The magnetization of a disturbing body may be induced by the earth's magnetic field and it may be remanent (permanent). In order to make the formulae more generally binding, we derive them for the case where the edge forms an angle with y-axis, i. e. the angles α and γ are $\neq 0$.

The parts of $M_{/\!/}$ and $M_\perp$ that are induced by the earth's field may be expressed as

$$\varkappa'_{/\!/} T' \cos(i' - \varphi') \text{ and } \varkappa'_\perp T' \sin(i' - \varphi').$$

For the remanent magnetization the corresponding expressions are

$$\frac{I_r' \cos(\psi' - \varphi')}{1 + \varkappa N_{/\!/}} = \varkappa'_{/\!/} k' T' \cos(\psi' - \varphi') \text{ and}$$

$$\frac{I_r' \sin(\psi' - \varphi')}{1 + \varkappa N_\perp} = \varkappa'_\perp k' T' \sin(\psi' - \varphi').$$

Thus we may write

$$\left.\begin{aligned} M_{/\!/} &= \varkappa'_{/\!/} T' [\cos(i' - \varphi') + k' \cos(\psi' - \varphi')] = \varkappa'_{/\!/} T'' \cos(i'' - \varphi'), \\ M_\perp &= \varkappa'_\perp T' [\sin(i' - \varphi') + k' \sin(\psi' - \varphi')] = \varkappa'_\perp T'' \sin(i'' - \varphi'). \end{aligned}\right\} \ldots \quad (3.1)$$

The dashed (′ or″) quantities may be calculated with the aid of the following formulae:

$$\left.\begin{aligned}
&T' = T \sin v_i, \text{ where } \cos v_i = \sin \gamma \sin i - \cos \gamma \cos i \sin (\alpha - \vartheta),\\
&\tan i' = - \frac{\sin \gamma \sin (\alpha - \vartheta) + \cos \gamma \tan i}{\cos (\alpha - \vartheta)},\\
&I_r' = I_r \sin v_r, \text{ where } \cos v_r = \sin \gamma \sin \psi - \cos \gamma \cos \psi \sin (\alpha - \vartheta_r),\\
&\tan \psi' = - \frac{\sin \gamma \sin (\alpha - \vartheta_r) + \cos \gamma \tan \psi}{\cos (\alpha - \vartheta_r)},\\
&\cos \varphi' = \frac{\cos \varphi}{\cos \gamma}, \quad k' = k \frac{\sin v_r}{\sin v_i}, \quad \varkappa'_{//} = \frac{\varkappa}{1 + \varkappa N_{//}}, \quad \varkappa'_{//} = \frac{\varkappa}{1 + \varkappa N_{\perp}},\\
&T'' = T' \sqrt{1 + 2k' \cos (i' - \psi') + k'^2}, \quad \tan i'' = \frac{\sin i' + k' \sin \psi'}{\cos i' + k' \cos \psi'}.
\end{aligned}\right\} \quad (3.2)$$

As regards the demagnetizing factors $N_{//}$ and $N_{\perp}$, they are strictly speaking not constant within a sheet-like body. This is especially the case at the edges. Only for bodies limited by faces of second degree, the demagnetizing factors are invariable inside the bounding surfaces. On the other hand, the sum of the demagnetizing factors available for three mutual perpendicular directions is constant ($= 4\pi$) for every point of a body of any shape. Thus we have

$$N_{=} + N_{//} + N_{\perp} = 4\pi.$$

Starting from this relation and the assumption that the "mean magnitudes" of $N_{=}$, $N_{//}$, $N_{\perp}$ are in a direct ratio to the areas of sections of the sheet-like body in planes perpendicular to the three corresponding directions of demagnetizing, we arrive at the formulae

$$N_{=} = 4\pi \frac{\varepsilon d}{\varepsilon d + 2\varepsilon l + 2dl}, \quad N_{//} = 4\pi \frac{2\varepsilon l}{\varepsilon d + 2\varepsilon l + 2dl},$$

$$N_{\perp} = 4\pi \frac{2\,dl}{\varepsilon d + 2\,\varepsilon l + 2\,dl} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad)$$

For an infinite length of the edge ($l = \infty$) these formulae give

$$N_{=} = 0, \quad N_{//} = 4\pi \frac{\varepsilon}{\varepsilon + d}, \quad N_{\perp} = 4\pi \frac{d}{\varepsilon + d}.$$

When even the extension in the depth of the sheet is infinite ($d = \infty$), we obtain

$$N_{=} = N_{//} = 0, \quad N_{\perp} = 4\pi.$$

The magnetic susceptibility of rocks is primarily determined by their content of ferromagnetic minerals. Of the common minerals only magnetite, pyrrhotite and hematite in α-form (maghemite) are ferromagnetic. Pyrrhotite, however, is not seldom paramagnetic.

Pyrrhotite and in particular magnetite occur to a very great extent as accessory minerals in the rocks, and have for that reason a dominating influence on

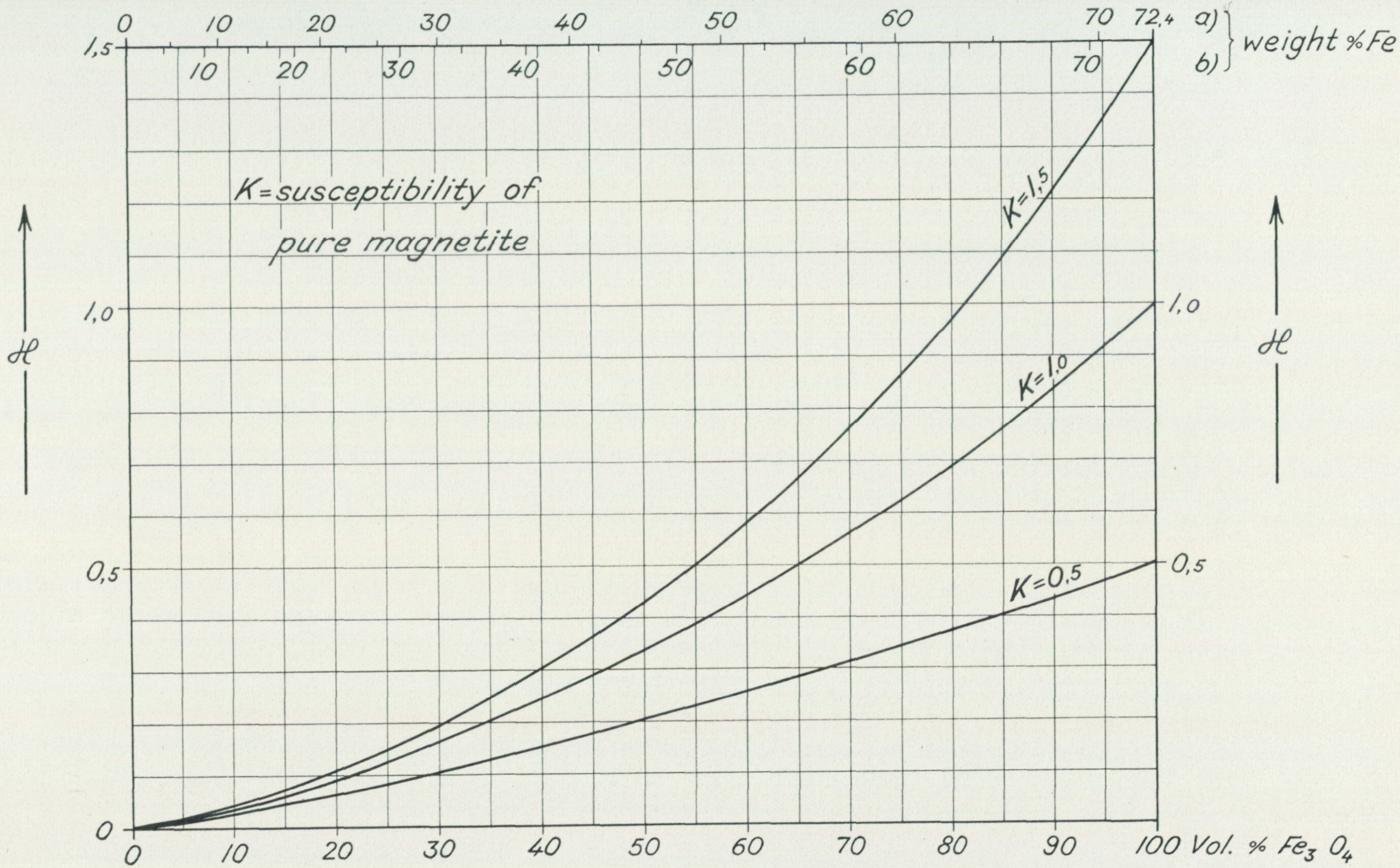

Fig. 3.1. The relation between susceptibility ($\varkappa$) and the volume percentage magnetite (Fe_3O_4) in ores and rocks. The curves correspond to the values 0.5, 1.0 and 1.5 of the susceptibility (K) of pure magnetite. The scale divisions a) and b) give the weight percentages of F_e, the gangue material having a specific gravity of 3.17 and 2.67 respectively.

the magnetic properties of the rocks. Crystals of these minerals are magnetically anisotropic, and show remarkable differences in susceptibility in different directions. Samples of rocks and ores have, however, usually the same susceptibility in all directions, due to the completely arbitrary orientation of the occurring ferromagnetic minerals as regards their crystal axes (pseudoisotropy).

As far as the magnetic properties of rocks and ores are concerned, the susceptibility at field strength of the magnitude of the earth's field varies from about 0.5 to 2 for magnetite, and from about 0.05 to 0.2 for ferromagnetic pyrrhotite.

According to investigations carried out on rocks and ores within the Swedish iron-ore fields (23), the susceptibility of the material in a body containing v volume fraction of ferromagnetic material of the susceptibility (K), is

$$\varkappa = \frac{K\mathrm{v}}{1 + CK\mathrm{v}}, \text{ where } C = \text{internal demagnetizing factor} = \tfrac{4}{3}\pi \frac{1 - \mathrm{v}^{1/6}}{\mathrm{v}}.$$

The relation between $\varkappa$ and the volume percentage of Fe_3O_4 in a magnetite ore is shown in the diagram in Fig. 3.1. The three curves hold for the cases in which the susceptibility (K) of pure magnetite are 1.5, 1.0 and 0.5 respectively. The largest K-value is representative of typical skarn-bearing and calcareous iron ores. Quartzeous iron ores have a K-value of about 1. Apatitic iron ores and manganiferrous iron ores have values of K varying greatly from deposit to deposit. The highest values are about 1.5, and the lowest values are about 0.5.

In the diagram there is also shown the weight percentage of Fe corresponding to the volume percentage of Fe_3O_4 for those cases in which the gangue material has a specific gravity of a) 3.17 and b) 2.67; the specific gravity of magnetite is fixed to 5.17. The former value (3.17) corresponds approximately to green skarn, tremolite, anthophyllite, hornblende and apatite, while the latter value (2.67) corresponds to acid rocks.

Judging from the experience gained from data in published cases of interpretation of local magnetic anomalies, it appears, that the natural remanence in the majority of the cases considered must either be small relative to the induced magnetization or else have approximatively the same direction as the latter. Exceptions to this rule, however, are not uncommon.

§ 4. Tables.

When drawing up auxiliary tables for calculating X and Z according to the formulae (2.8—2.9) it is convenient to separate the cases of l infinite and l finite and further $t \neq 0$ and $t = 0$.

In the following the notations below are used.

$$\varepsilon' = \frac{\varepsilon}{t}; \quad x' = \frac{x}{t}; \quad l' = \frac{l}{t}; \quad C' = \sqrt{\frac{1 + l'^2}{1 + x'^2 + l'^2}};$$

$$k_t' = \frac{C'}{1 + x'^2}; \quad k_x' = x' k_t'; \quad C_\perp' = \frac{C'}{\sin\varphi + x' \cos\varphi \overset{+}{(-)} \sqrt{1 + x'^2 + l'^2}};$$

$$\varepsilon'' = \frac{\varepsilon}{l}; \quad x'' = \frac{x}{l}; \quad k_x'' = \frac{1}{x''\sqrt{1 + x''^2}};$$

$$C_\perp'' = \frac{k_x''}{\cos\varphi + k_x''(1 + x''^2)};$$

$A = 2\varepsilon l$ = area of upper, or lower edge of sheet.

In the expression for $C_\perp'$ as also in the formulae given later, the sign inside the brackets is used when t is negative. The fundamental formulae (2.8—2.9) for the cases set out above may now be expressed as follows

l infinite

$$t \neq 0 \quad \begin{cases} X = -2\varepsilon' [k_x' M_{//} + k_t' M_\perp] \quad (4.1\,a) \\ Z = 2\varepsilon' [k_t' M_{//} - k_x' M_\perp] \quad (4.1\,b) \end{cases}$$

$$t = 0 \quad X = -\frac{2\varepsilon M_{//}}{x} \quad (4.2\,a) \qquad Z = \frac{2\varepsilon M_\perp}{x} \quad (4.2\,b)$$

l finite

$$t \neq 0 \quad \begin{cases} X = \underset{(+)}{-}\, 2\varepsilon' \dfrac{l'}{\sqrt{1 + l'^2}} [k_x' M_{//} + (k_t' - C_\perp' \sin\varphi)\, M_\perp] \quad (4.3\,a) \\ Z = \underset{(-)}{+}\, 2\varepsilon' \dfrac{l'}{\sqrt{1 + l'^2}} [k_t' M_{//} - (k_x' - C_\perp' \cos\varphi)\, M_\perp] \quad (4.3\,b) \end{cases}$$

$$t = 0 \begin{cases} X = -2\varepsilon'' [k_x'' M_{//} - C_\perp'' \sin\varphi\, M_\perp) \quad (4.4\,a) \\ Z = -2\varepsilon'' M_\perp (k_x'' - C_\perp'' \cos\varphi) \quad (4.4\,b) \end{cases}$$

The factor $2\varepsilon' \dfrac{l'}{\sqrt{1 + l'^2}}$ may be replaced by $\dfrac{A}{t^2\sqrt{1 + l'^2}}$.

The equations (4.3 *a*) and (4.3 *b*) may also be applied to rod-like bodies. In that case A denotes the area of the cross section of the rod-shaped body perpendicular to its longitudinal axis. A qualification is, however, that the area can be inscribed in a rectangle of sides $2l'$ and ε' where both these sides are $\leqslant 1$. The formulae will now express the values of X and Z in a vertical plane through the longitudinal axis of the body.

Tables for these cases have been compiled and will be found on pages (I—XXI) in the end of this paper. Table 1 applies to the case of l infinite and $t \neq 0$, table 2 to l finite and $t \neq 0$ and table 3 to l finite and $t = 0$.

Table 1 includes values of k_t' and k_x'. Since in this case, $C' = 1$ these quantities are simply functions of x'. k_t' is always positive, whereas k_x' has the same sign as x'. The numerical values of k_t' and k_x' do not change when x' changes sign.

Table 2 embraces a number of double pages for different values of l' (l' = 0, 1, 2, 3, 4, 5, 6, 8 and 10) where k_t', k_x', $C_\perp'$ and $C_\perp' \cos\varphi$ are tabulated for close successive values of x' from 0 to 10 and from 0 to — 10. The left-hand pages contain values that correspond to positive values of x' while the right-hand pages correspond to negative values of x'. k_t' and k_x' are entered only in the left-hand pages, as these quantities do not alter their numerical values as x' changes sign. k_t' is always positive, whereas k_x' is to have the same sign as x'. This rule holds whether t is positive or negative.

$C_{\perp}'$ has always the same sign as t, but changes its numerical value as t changes sign. In the table the values of $C_{\perp}'$ and $C_{\perp}' \cos \varphi$ correspond to positive values of t. $C_{\perp}'$ has only been tabulated for $\varphi = 90°$ and $270°$ ($-90°$), and it remains unchanged for both angles when x' changes sign. $C_{\perp}'$ is thus included only in the right-hand pages of the table in the two columns farthest to the right. When t is negative the values in the column of $\varphi = 90°$ are valid for $-C_{\perp}'$ with $\varphi = -90°$, and the column of $\varphi = -90°$ is valid for $-C_{\perp}'$ with $\varphi = +90°$.

When magnetic measurements in the field are carried out, only Z is usually measured. It has thus been considered convenient to tabulate the values of $C_{\perp}' \cos \varphi$ for close successive steps of φ (10 degree intervals from 0—80°). If $C_{\perp}' \sin \varphi$ is required multiply the tabulated value by $\tan \varphi$.

If one has to calculate the components of X and Z along a profile line passing over the middle of the plate perpendicular to the edge, it is always possible to place the coordinate system xyz (see Fig. 2.1) in such a manner that the x-axis gets parallel to the profile line, and the dip-angle φ of the plate lies between 0° and 90°. Then the values of $C_{\perp}' \cos \varphi$ in table 2 are valid if the profile line cuts the negative z-axis (t positive) and thus passes over the plate. If, on the contrary, the profile line cuts the positive z-axis (t negative), and thus passes through the plate one cannot use table 2, since $C_{\perp}'$ changes its numerical value if t changes its sign.

Table 3. The numerical value of k_x'' does not change when x'' changes sign. k_x'' is thus entered only in the left-hand pages of the table. On the other hand k_x'' is positive when x'' is positive and negative when x'' is negative.

$C_{\perp}''$ is always positive. $C_{\perp}'' \cos \varphi$ is hence positive if φ is situated in the 1st or 4th quadrant, and negative if in the 2nd or 3rd quadrant. From the expression for $C_{\perp}''$ it is clear that this quantity has the same numerical value at a point x'' with $\varphi = \varphi_{\mathrm{I}}$ or φ_{IV} in the 1st resp. 4th quadrant or at a point $-x''$ with $\varphi = (180° - \varphi_{\mathrm{I}}) = \varphi_{\mathrm{II}}$ or $\varphi = (270° - \varphi_{\mathrm{I}}) = \varphi_{\mathrm{III}}$ in the 2nd or 3rd quadrant. The tabulated values of $C_{\perp}'' \cos \varphi$ for $\varphi = \varphi_{\mathrm{I}}$ are hence also valid for $\varphi = \varphi_{\mathrm{IV}}$ without modifications. If $\varphi = \varphi_{\mathrm{II}}$ or φ_{III} the right-hand pages of the table are to be used for positive x''-values and the left-hand pages for negative x''-values.

CHAPTER II.

Methods of Interpretation. Single Sheet of Infinite Length.

§ 5. Introduction.

In this chapter and in the following that case is treated where the disturbing body may be regarded as consisting of a thin sheet of infinite extent in depth. As mentioned earlier in § 2 a sheet of finite extent in depth is from a magnetic point of view equivalent to two sheets of infinite depth. This case is treated in chapter V.

Suppose that the horizontal and vertical intensities are known at a number of points along a horizontal orthogonal profile line, and that the sheet edge is horizontal and of infinite length. In Fig. 2.1 the above mentioned profile line corresponds to a line ($z = -t = -t_0$; $y = 0$;) parallel to the x-axis, and the given field data correspond to X and Z in formulae (2.9). These formulae, however, presuppose that the x-coordinate of the head point at the profile line is zero.

When interpreting a magnetic profile curve corresponding to a profile line, the position of the head point is mostly unknown. Let us therefore suppose that origo of the coordinate system in Fig. 2.1 is displaced at an arbitrary distance along the x-axis and that after this displacement the x-axis cuts the edge at a point $x = x_0$. In order for the equations (2.9) to be valid in this case it is obviously necessary to substitute $(x - x_0)$ for x and write these equations

$$X = -2\frac{\varepsilon M_{//}(x - x_0) + \varepsilon M_{\perp} t}{t^2 + (x - x_0)^2}, \quad (5.1) \qquad Z = 2\frac{\varepsilon M_{//} t - \varepsilon M_{\perp}(x - x_0)}{t^2 + (x - x_0)^2} \,.. \quad (5.2)$$

According to the formulae above the components X and Z along a horizontal orthogonal profile line are governed by four parameters, viz. x_0, t_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$. In the general case where the edge slopes and the profile line is not horizontal and orthogonal, the components X and Z along the profile line (situated in the xz-plane) must be governed also by the angles α, β and γ of which β is known. In the following the quantities x_0, t_0, α, γ, $\varepsilon M_{//}$, $\varepsilon M_{\perp}$ are called the *magnetic parameters* of a sheet. Of those the four first written quantities determine the position of the edge and the last two state the magnetization of the sheet.

The parameters x_0 and t_0 are, exactly expressed, the coordinates of the *magnetic* edge line (the *line pole*) of the sheet. This line pole does not always coincide with the edge of the sheet-like body, but may in many cases be situated within the body at some distance from its edge.

According to equations (3.1), $M_{//}$ and $M_{\perp}$ are functions of the magnetic properties of the sheet material (susceptibility, natural remanence) as well as of the position of the sheet in relation to the direction of the earth's normal field. The values of $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ may therefore form the basis for further calculations of the position of the sheet (dip angle) as well as for estimating the magnetic properties and thickness of the sheet.

From the above it is clear that the interpretation calculations fall into two main stages, viz. a) the determination of the magnetic parameters of the sheet, and b) calculations based on $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$. The calculations according to a) aim at a mathematical representation of the given profile curves, while the calculations according to b) imply an attempt to explain the magnetization of the sheet. We start with the primary problem a). The secondary problem b) is treated later on in § 8.

As regards the determination of the magnetic parameters of a sheet, the methods of calculation given in this paper are based on the use of intensity values of field components at *datum points*, situated at *one reference line*. Thereby it is most convenient to choose the reference line in such a manner as for it to be situated in a plane perpendicular to the edge. When the results of the magnetic measurements in the terrain are represented on maps where the isodynamic lines are drawn up in the recognized manner, this desirable object implies, from a practical point of view, that the data used for the calculation ought to be taken from the map along a line running at right angles to the centre line of the anomaly image.

It is convenient to represent the data, taken along a profile line on a map in the manner mentioned above, in a diagram (in the following denoted *interpretation diagram*) according to that in Fig. 5.1. The full line in this diagram represents the *magnetic profile curve* of the component F, measured along the shaded ground profile, and L is the reference line used in the calculations. The points 1, 2, 3, 4 on the profile curve of F correspond to the points where the reference line cuts the ground profile, i. e. the *datum* points of the reference line. Thus we obtain from the diagram the data x_1, F_1 for the datum point (1), the data x_2, F_2 for the datum point (2) and so on.

The axes of length (x) and height (h) in the interpretation diagram are drawn on the same scale, so that the slope angle (β) of the reference line may be taken directly from the diagram.

As regards all the quantities (as coordinates, field components, angles and so on) represented in the interpretation diagram, they are connected with the definitions of the symbols in § 1 by the stipulation that the plane of the diagram is the xz-plane through origo of our coordinate-system xyz, and that the yz-plane through origo cuts the length-axis (x) of the diagram perpendicular and at the chosen zero point. Further we assume that the x-axis in the diagram is positive in the same direction as the x-axis of the coordinate system xyz.

By treating the methods of calculation as regards the magnetic parameters of a sheet, we distinguish in the following between *simple cases* and *general cases*. When the edge is horizontal and the reference line is horizontal and orthogonal, i. e. $\alpha = \beta = \gamma = 0°$, we speak of simple cases and otherwise we speak of general cases.

§ 6. Simple cases.

As $\alpha = \beta = \gamma = 0$, the formulae (5.1) and (5.2) may be used in the calculation of the magnetic parameters x_0, t_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ of the sheet. As there are

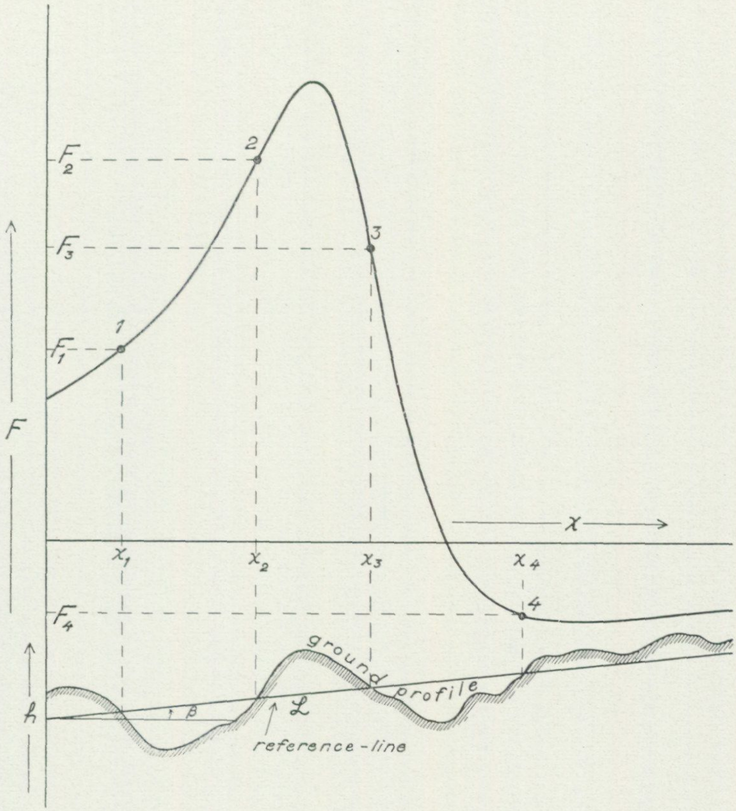

Fig. 5.1. Interpretation diagram showing a profile curve of F, the ground profile, the reference-line (L), and the calculation points.

four parameters, four equations are required for the calculations. Thus it is only necessary to know X or Z at four points, or X and Z at two points on the reference line.

Given X or Z.

Equations (5.1) and (5.2) may both be written in the following form

$$a_0 + a_1 x + b_0 F + b_1 x F = x^2 F \quad (6.1)$$

where, if $F = X$

$$a_0 = 2\,(\varepsilon M_{//}\, x_0 - \varepsilon M_{\perp}\, t_0), \quad b_0 = -\,t_0^2 - x_0^2,$$
$$a_1 = -\,2\,\varepsilon M_{//}, \quad b_1 = 2\,x_0,$$

and if $F = Z$ $\quad (6.2)$

$$a_0 = 2\,(\varepsilon M_{//}\, t_0 + \varepsilon M_{\perp}\, x_0), \quad b_0 = -\,t_0^2 - x_0^2,$$
$$a_1 = -\,2\,\varepsilon M_{\perp}, \quad b_1 = 2\,x_0.$$

Using the values of x_i and F_i at four datum points (corresponding to $i = 1\,..\,4$) on a magnetic profile curve of X or Z, we obtain four linear equations according to (6.1) for calculating the parameters a_0, a_1, b_0 and b_1.

Putting

$$\left.\begin{aligned} D &= \left| \begin{matrix} 1 & x_i & F_i & x_i F_i \end{matrix} \right|, \\ D_{a_0} &= \left| \begin{matrix} x_i^2 F_i & x_i & F_i & x_i F_i \end{matrix} \right|, \\ D_{a_1} &= \left| \begin{matrix} 1 & x_i^2 F_i & F_i & x_i F_i \end{matrix} \right|, \\ D_{b_0} &= \left| \begin{matrix} 1 & x_i & x_i^2 F_i & x_i F_i \end{matrix} \right|, \\ D_{b_1} &= \left| \begin{matrix} 1 & x_i & F_i & x_i^2 F_i \end{matrix} \right|, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6.3)$$

where the expressions with the vertical lines are determinants of four rows (the first row corresponding to $i = 1$, the second to $i = 2$ and so on), we have

$$a_0 = \frac{D_{a_0}}{D}, \; a_1 = \frac{D_{a_1}}{D}, \; b_0 = \frac{D_{b_0}}{D}, \; b_1 = \frac{D_{b_1}}{D} \quad \ldots\ldots\ldots\ldots \quad (6.4)$$

Further we obtain from the expression (6.2)

$$x_0 = \tfrac{1}{2} b_1,$$
$$t_0 = \pm \tfrac{1}{2} \sqrt{-4 b_0 - b_1^2},$$

and if $F = X$

$$\varepsilon M_{\perp} = - \frac{2 a_0 + a_1 b_1}{4 t_0},$$
$$\varepsilon M_{//} = - \tfrac{1}{2} a_1, \quad \ldots\ldots\ldots\ldots \quad (6.5)$$

and if $F = Z$

$$\varepsilon M_{//} = \frac{2 a_0 + a_1 b_1}{4 t_0},$$
$$\varepsilon M_{\perp} = - \tfrac{1}{2} a_1.$$

It should be observed that in the above cases the numerical values of the magnetic parameters of the sheet are unambiguously fixed except for the sign of two of them. When using the profile curve of X the signs of t_0 and of $\varepsilon M_{\perp}$ will be uncertain but that of the product $\varepsilon M_{//} t_0$ will be fixed. When basing the calculations on a profile curve for Z, a similar situation applies to t_0 and $\varepsilon M_{//}$. Hence it is not possible to decide solely on calculations whether the edge of the sheet is situated above or below the reference line, if the values of only one of the field components X or Z is known along its length. When applied to actual examples, ambiguity arises only at interpretations of measurements below the earth's surface.

In the following the phrase *interpretation eqvation* is applied to equation (6.1) and the phrase *parameters of the interpretation eqvation* to a_0, a_1, b_0 and b_1. As these quantities are functions of the magnetic parameters of the sheet, they are also called *parameter functions.*

GIVEN F.

The interpretation method described above may be extended to a component F. As the direction cosines of F are v_x, v_y and v_z, we may write

$$F = v_x X + v_y Y + v_z Z,$$

where $Y = 0$. If the interpretation equations for X and Z be multiplied by v_x and v_z respectively and added, it is easily seen that the resulting equation may be expressed in the same form as (6.1). The expressions for b_0 and b_1 are in this case the same as in (6.2) and

$$\left.\begin{aligned} a_0 &= 2\,\varepsilon M_{//}(x_0 v_x + t_0 v_z) + 2\,\varepsilon M_{\perp}(x_0 v_z - t_0 v_x) \\ a_1 &= 2\,\varepsilon M_{//} v_x - 2\,\varepsilon M_{\perp} v_z \end{aligned}\right\} \qquad (6.6)$$

Thus we obtain

$$\left.\begin{aligned} x_0 &= \tfrac{1}{2} b_0, \\ t_0 &= \pm\sqrt{-4\,b_0 - b_1^2}, \\ \varepsilon M_{//} &= \frac{(2a_0 + a_1 b_1)\ v_z - 2\,a_1 t_0 v_x}{4\,t_0\,(v_x^2 + v_z^2)}, \\ \varepsilon M_{\perp} &= -\frac{(2\,a_0 + a_1 b_1)\,v_x + 2\,a_1 t_0 v_z}{4\,t_0\,(v_x^2 + v_z^2)}. \end{aligned}\right\} \qquad (6.7)$$

The calculations of the magnetic parameters of a sheet where we know a component F can thus be done in the same way as in the cases when the component X or the component Z is given. The first step is to compute the values of the parameter functions a_0, a_1, b_0, b_1 according to the expressions (6.3) and (6.4) and the second step to use these values to compute x_0, t_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ with the aid of the formulae (6.7).

When $F = X$ we have $v_x = 1$, $v_z = 0$ and when $F = Z$, we have $v_x = 0$ and $v_z = 1$. As is easily verified the equations (6.6) and (6.7) revert in these cases to the corresponding expressions in (6.2) and (6.5).

Given X and Z.

On solving the parameters $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ from equations (5.1, 5.2) we obtain

$$\left.\begin{aligned} 2\,\varepsilon M_{//} &= -(x - x_0)\,X + t\,Z, \\ 2\,\varepsilon M_{\perp} &= -t\,X - (x - x_0)\,Z. \end{aligned}\right\} \qquad (6.8)$$

For two datum points (1 and 2) on a horizontal profile line we thus obtain

$$\left.\begin{aligned} -(x_1 - x_0)\,X_1 + t_0 Z_1 &= -(x_2 - x_0)\ X_2 + t_0 Z_2, \\ t_0 X_1 + (x_1 - x_0)\ Z_1 &= t_0 X_2 + (x_2 - x_0)\ Z_2. \end{aligned}\right\}$$

x_0 and t_0 may be solved from the two above equations and then $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ are immediately obtainable from equations (6.8). The following expressions for the parameters are obtained

$$\left.\begin{aligned} x_0 &= \frac{(X_1 - X_2)\,(x_1 X_1 - x_2 X_2) + (Z_1 - Z_2)\,(x_1 Z_1 - x_2 Z_2)}{(X_1 - X_2)^2 + (Z_1 - Z_2)^2}, \\ t_0 &= \frac{(x_2 - x_1)\,(X_1 Z_2 - X_2 Z_1)}{(X_1 - X_2)^2 + (Z_1 - Z_2)^2}, \\ 2\,\varepsilon M_{//} &= \frac{(x_2 - x_1)\,[X_1\,(X_2^2 + Z_2^2) - X_2\,(X_1^2 + Z_1^2)]}{(X_1 - X_2)^2 + (Z_1 - Z_2)^2}, \\ 2\,\varepsilon M_{\perp} &= \frac{(x_2 - x_1)\,[Z_1\,(X_2^2 + Z_2^2) - Z_2\,(X_1^2 + Z_1^2)]}{(X_1 - X_2)^2 + (Z_1 - Z_2)^2}. \end{aligned}\right\} \qquad (6.9)$$

The magnetic parameters of the sheet are thus unambiguously fixed in this case.

It exists a simple relation between the values of X and Z for the points on the profile line. To show this in a more general way we assume that the components of the anomaly field of the sheet are $X - \Delta X$ and $Z - \Delta Z$, where ΔX and ΔZ are constant quantities of correction of the given X- and Z-values.

According to the equations (6.8) we may write

$$\left.\begin{aligned} 2\,\varepsilon M_{/\!/} &= -(x - x_0)(X - \Delta X) + t_0(Z - \Delta Z),\\ 2\,\varepsilon M_{\perp} &= -(x - x_0)(Z - \Delta Z) - t_0(X - \Delta X).\end{aligned}\right\}$$

Putting $X = X_0$ and $Z = Z_0$ in the head point $x = x_0$ we obtain

$$2\,\varepsilon M_{/\!/} = t_0(Z_0 - \Delta Z), \quad 2\,\varepsilon M_{\perp} = -\,t_0(X_0 - \Delta X).$$

Inserting these expressions in the above equation we find

$$\frac{t_0}{x - x_0} = \frac{X - \Delta X}{Z - Z_0} = -\frac{Z - \Delta Z}{X - X_0},$$

and arrive to the relation

$$(X - \Delta X)(X - X_0) + (Z - \Delta Z)(Z - Z_0) = 0.$$

This expression may be written

$$\left.\begin{aligned} &(X - a)^2 + (Z - b)^2 = r^2,\\ \text{where}\quad &a = \tfrac{1}{2}(X_0 + \Delta X),\\ &b = \tfrac{1}{2}(Z_0 + \Delta Z),\\ &r^2 = a^2 + b^2 + X_0\,\Delta X - Z_0\,\Delta Z.\end{aligned}\right| \quad\ldots\ldots\ldots\ldots \quad (6.10)$$

On using the relations

$$X_0 = -2\frac{\varepsilon M_{\perp}}{t_0} + \Delta X, \quad Z_0 = \frac{2\,\varepsilon M_{/\!/}}{t_0} + \Delta Z,$$

the expressions (6.10) may be transformed into

$$\left.\begin{aligned} a &= -\frac{\varepsilon M_{\perp}}{t_0} + \Delta X,\\ b &= \frac{\varepsilon M_{/\!/}}{t_0} + \Delta Z,\\ r &= \frac{\varepsilon M_{T}}{t_0}.\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots \quad (6.11)$$

Thus, if the X- and Z-values at every point on the profile line are represented by a point in an XZ-diagram, the profile curves of X and Z correspond in this diagram to a circle. The centre of this circle has the coordinates $X = a$ and $Z = b$, and the radius is r.

§ 7. General cases.

Let us assume two coordinate systems (x, y, z) and (x', y', z') with a common origin. Further, that the xy-plane and the x'-axis are horizontal, and that a sheet is orientated in the latter system as in Fig. 2.1. In addition, we assume that the corresponding positive axes of the two systems form acute angles.

The coordinate system $x'y'z'$ can be transformed into the xyz-system by turning the system first an acute angle $(-\gamma)$ around the x-axis and after that an acute angle $(-\alpha)$ around the z'-axis. We reckon these angles in relation to the xyz-system, and in agreement with the definitions in § 1. Then the following relationships hold between the field components in the two coordinate systems

$$\left.\begin{aligned} X &= X' \cos\alpha - Y' \sin\alpha \cos\gamma + Z' \sin\alpha \sin\gamma, \\ Y &= X' \sin\alpha + Y' \cos\alpha \cos\gamma - Z' \cos\alpha \sin\gamma, \\ Z &= Y' \sin\gamma + Z' \cos\gamma. \end{aligned}\right\} \quad \ldots\ldots (7.1)$$

As the sheet is assumed to have infinite length $Y' = 0$. The coordinates of a point are determined by equations which are completely analogous to those shown in (7.1). For the following, however, direct expressions for x', y' and z' are required. These are

$$\left.\begin{aligned} x' &= x \cos\alpha + y \sin\alpha, \\ y' &= - x \sin\alpha \cos\gamma + y \cos\alpha \cos\gamma + z \sin\gamma, \\ z' &= x \sin\alpha \sin\gamma - y \cos\alpha \sin\gamma - z \cos\gamma. \end{aligned}\right\} \ldots\ldots (7.2)$$

Suppose that the intensity F is known at some points situated on a sloping profile line in the xz-plane through the origin. If the point of intersection of the profile line with the z-axis (head point) is denoted by $z_0 = - t_0$ it is obvious that the equation of the line is

$$z = - t = - t_0 - x \tan \beta.$$

If the origin is moved along the x-axis to the point $x = x_0$, x is to be replaced by $(x - x_0)$ in the expression above. Since the chosen profile is defined so that $y = 0$, we obtain, according to (7.2).

$$\begin{aligned} x' &= (x - x_0) \cos \alpha, \\ z' &= - t' = (x - x_0) \sin \alpha \sin \gamma - \cos\gamma\, [t_0 + (x - x_0) \tan \beta]. \end{aligned} \quad \ldots\ldots (7.3\ a)$$

The latter equation may also be written

$$\left.\begin{aligned} t' &= \cos \gamma\, [t_0 + p(x - x_0)], \\ \text{where} \quad p &= \tan\beta - \sin\alpha \tan \gamma. \end{aligned}\right\} \ldots\ldots\ldots\ldots (7.3\ b)$$

With the aid of the equations (7.1), we obtain

$$\left.\begin{aligned} F &= v_x X + v_y Y + v_z Z \\ &= X' (v_x \cos\alpha + v_y \sin\alpha) + Z'(v_x \sin\alpha \sin\gamma - v_y \cos\alpha \sin\gamma + v_z \cos\gamma) \\ &= X' v_x' + Z' v_z'. \\ \text{where } v_x' &= v_x \cos\alpha + v_y \sin\alpha, \\ v_z' &= v_x \sin\alpha \sin\gamma - v_y \cos\alpha \sin\gamma + v_z \cos\gamma, \end{aligned}\right\} \ldots (7.4)$$

are the direction cosines of F referred to the x' - and z'-axes.

Equations (5.1) and (5.2) are clearly valid for X', Z', x' and t'. As $x_0' = 0$ the equation (7.4) may be transformed into

$$2\,t'\,(\varepsilon M_{//} v'_z - \varepsilon M_{\perp} v'_x) - 2\,x'\,(\varepsilon M_{//} v'_x + \varepsilon M_{\perp} v'_z) - t'^2 F = x'^2 F \quad \ldots\ldots \quad (7.5)$$

In the following we use the equations (7.3), (7.4) and (7.5) by generalising the interpretation methods earlier treated.

Given one component (F).

If we replace t' and x' by the expressions in (7.3 b and a), the equation (7.5) assumes the form of the interpretation equation, if we put

$$\left.\begin{aligned}
a_0 &= 2\,\frac{x_0\,(\varepsilon M_{//} v_x' + \varepsilon M_{\perp} v_z')\cos\alpha + (t_0 - p x_0)\,(\varepsilon M_{//} v_z' - \varepsilon M_{\perp} v_x')\cos\gamma}{\mu^2 \cos\alpha \cos\gamma},\\
a_1 &= -\,2\,\frac{(\varepsilon M_{//} v_x' + \varepsilon M_{\perp} v_z')\cos\alpha - p\,(\varepsilon M_{//} v_z' - \varepsilon M_{\perp} v_x')\cos\gamma}{\mu^2 \cos\alpha \cos\gamma},\\
b_0 &= -\,\frac{x_0^2 \cos^2\alpha + (t_0 - p x_0)^2 \cos^2\gamma}{\mu^2 \cos\alpha \cos\gamma},\\
b_1 &= 2\,\frac{x_0 \cos^2\alpha - p\,(t_0 - p x_0)\cos^2\gamma}{\mu^2 \cos\alpha \cos\gamma},\\
\text{where}\quad \mu^2 &= \frac{\cos^2\alpha + p^2\cos^2\gamma}{\cos\alpha\cos\gamma}, \quad p = \tan\beta - \sin\alpha\,\tan\gamma,\\
v_x' &= v_x \cos\alpha + v_y \sin\alpha,\\
v_z' &= v_x \sin\alpha \sin\gamma - v_y \cos\alpha \sin\gamma + v_z \cos\gamma.
\end{aligned}\right\} \quad \ldots\ldots \quad (7.6)$$

From equations (7.6) we obtain

$$\left.\begin{aligned}
t_0 &= \pm\,\frac{\mu^2}{2}\sqrt{-4\,b_0 - b_1^2},\\
x_0 &= \frac{b_1}{2} + \frac{p\cos\gamma}{\mu^2\cos\alpha}\,t_0,\\
\varepsilon M_{//} &= \frac{\mu^2\,(2\,a_0 + a_1 b_1)\,(v_z'\cos\alpha + p v_x'\cos\gamma) - 2\,a_1 t_0\,(v_x'\cos\alpha - p v_z'\cos\gamma)}{4\,t_0\,(v_x'^2 + v_z'^2)},\\
\varepsilon M_{\perp} &= -\,\frac{\mu^2\,(2\,a_0 + a_1 b_1)\,(v_x'\cos\alpha - p v_z'\cos\gamma) + 2\,a_1 t_0\,(v_z'\cos\alpha + p v_x'\cos\gamma)}{4\,t_0\,(v_x'^2 + v_z'^2)}.
\end{aligned}\right\} \quad (7.7)$$

The formulae (7.6) indicate the general expressions for the parameter functions. μ^2 is always positive since α and γ are acute angles by definition. The expressions for b_0 and b_1 do not contain v'_x and v'_z. Thus for a given profile line these two parameter functions have the same numerical values no matter what direction is chosen for F.

According to (7.7) there are always two solutions to t_0. These have the same numerical value but are opposite in sign. For each of the two values of t_0 there corresponds a series of unambiguous values of x_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$. When interpreting measurements are carried out above ground, t_0 may generally be assumed to be positive. Hence in this case as well as in other cases where the sign of t_0 is known, the above mentioned parameters are all unambiguously determined.

In order to calculate x_0, t_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ from formulae (7.7) it is necessary to know, besides the parameter functions, the angles α and γ. The slope angle (β) of the reference line and the direction cosines v_x, v_y and v_z obviously can be considered as given quantities.

As regards α and γ, it is frequently possible to determine these angles direct and with sufficient accuracy from the anomaly picture in the present magnetic chart. Should it not be possible to do so, there exists the possibility of determining α and γ by treating the data from a second profile line on the magnetic chart in the way described above. If this second line is situated in a vertical plan parallel to the first line and if in this case the quantities are denoted with a dash ($'$), we may write

$$\tan\alpha = \frac{x_0 - x'_0}{y'}, \qquad \frac{\tan\gamma}{\cos\alpha} = \frac{(h_0 - h'_0) - (t_0 - t'_0)}{y'},$$

where h_0 and h'_0 are the elevations of the head points on the reference lines and y' is the y-coordinate for the vertical plane through the second line. With the aid of these equations we may calculate the angles α and γ.

Given two components (F and G).

We put

$$a = \frac{t_0}{\mu^2} \text{ and } b = x_0 - a\frac{p\cos\gamma}{\cos\alpha}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.8)$$

where μ^2 and p is given under (7.6).

Then we may according to the expressions (7.3) for a point x_1 write

$$\left.\begin{aligned} t_i' &= a\cos\alpha + (x_i - b)\,p\cos\gamma, \\ x_i' &= (x_i - b)\cos\alpha - ap\cos\gamma. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (7.9)$$

If we apply the equation (7.5) to the values F_i and G_i in the point x_i and further $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ are solved from these two equations, we obtain

$$\left.\begin{aligned} 2\varepsilon M_{//}(v_x'u_z' - v_z'u'_x) &= -F_1(t_i'u_x' + x_i'u_z') + G_i(t'_iv'_x + x_i'v_z') \\ &= a(A_vG_i - A_uF_i) - b(B_vG_i - B_uF_i) + \\ &\quad + x_i(B_vG_i - B_uF_i), \\ 2\varepsilon M_{\perp}(v_x'u_z' - v_z'u_x') &= -F_i(t_i'u_z' - x_i'u_x') + G_i(t'_iv_z' - x_i'v_x') \\ &= a(B_vG_i - B_uF_i) + b(A_vG_i - A_uF_i) - \\ &\quad - x_i(A_vG_i - A_uF_i), \\ \text{where } A_v = v_x'\cos\alpha - v_z'p\cos\gamma, &\quad B_v = v_x'p\cos\gamma + v_z'\cos\alpha, \\ A_u = u_x'\cos\alpha - u_z'p\cos\gamma, &\quad B_u = u_x'p\cos\gamma + u_z'\cos\alpha. \end{aligned}\right\} \ldots \quad (7.10)$$

For two points, corresponding to $i = 1$ and 2, the following formulae may be derived from the two equations above

$$\left.\begin{aligned} a &= \frac{t_0}{\mu^2} = \\ &= \frac{(v_x' u_z' - v_z' u_x')\,(x_2 - x_1)\,(F_1 G_2 - F_2 G_1)}{(u'^2_x + u'^2_z)\,(F_2 - F_1)^2 + (v'^2_x + v'^2_z)\,(G_2 - G_1)^2 - 2\,(v_x' u_x' + v_z' u_z')\,(F_2 - F_1)\,(G_2 - G_1)}, \\ b &= x_0 - a\,\frac{p \cos\gamma}{\cos\alpha} = \\ &= \frac{(u'^2_x + u'^2_z)\,(F_2 - F_1)\,(x_2 F_2 - x_1 F_1) + (v'^2_x + v'^2_z)\,(G_2 - G_1)\,(x_2 G_2 - x_1 G_1) -}{(u'^2_x + u'^2_z)\,(F_2 - F_1)^2 + (v'^2_x + v'^2_z)\,(G_2 - G_1)^2 -} \\ &\quad \frac{-(v_x' u_x' + v_z' u_z')\,[(F_2 - F_1)\,(x_2 G_2 - x_1 G_1) + (G_2 - G_1)\,(x_2 F_2 - x_1 F_1)]}{-2\,(v_x' u_x' + v_z' u_z')\,(F_2 - F_1)\,(G_2 - G_1)}. \end{aligned}\right\} \quad \ldots (7.11)$$

If (α) and (γ) are known, we may calculate v_x', v_z', u_x', u_z' and further p and μ^2 with the aid of the expressions (7.6) as v_x, v_y, v_z, u_x, u_y, u_z and (β) can obviously always be considered as known. In this case, thus x_0 and t_0 may be obtained from the above expressions (7.11). The quantities $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ may then be calculated from the equations (7.10), if in the latter are inserted the values of t'_i, x'_i, F_i and G_i that correspond to $i = 1$ or $i = 2$.

The relation between the two components F and G along a profile line may be represented graphically by a curve of second degree. To show this, we denote the values of F and G at the point $x = x_0$ on the datum line by F_0 and G_0. According to (7.10) we obtain

$$2\,\varepsilon M_{//}\,(v_x' u_z' - v_z' u_x') = t_0 \cos\gamma\,(-u_x' F_0 + v_x' G_0),$$

$$2\,\varepsilon M_{\perp}\,(v_x' u_z' - v_z' u_x') = t_0 \cos\gamma\,(-u_z' F_0 + v_z' G_0).$$

With the aid of these expressions we may arrive from (7.10) at the relations

$$\frac{x - x_0}{t_0 \cos\gamma} = \frac{u_x'\,(F - F_0) - v_x'\,(G - G_0)}{-F\,(p u_x' \cos\gamma + u_z' \cos\alpha) + G\,(p v_x' \cos\gamma + v_z' \cos\alpha)}$$

$$= \frac{u_z'\,(F - F_0) - v_z'\,(G - G_0)}{-F\,(p u_z' \cos\gamma - u_x' \cos\alpha) + G\,(p v_z' \cos\gamma - v_x' \cos\alpha)}.$$

From this we obtain

$$F\,(F - F_0)\,(u'^2_x + u'^2_z) + G\,(G - G_0)\,(v'^2_x + v'^2_z) + F\,(G - G_0)\left[p\,\frac{\cos\gamma}{\cos\alpha}\,(v_x' u_z' - v_z' u_x') - \right.$$

$$\left. -(v_x' u_x' + v_z' u_z')\right] - G\,(F - F_0)\left[p\,\frac{\cos\gamma}{\cos\alpha}\,(v_x' u_z' - v_z' u_x') + (v_x' u_x' + v_z' u_z')\right] = 0 \quad \ldots (7.12)$$

The above equation is a quadratic equation as regards F and G. Furthermore these components always have finite values, except when the datum line runs through the edge of the sheet. In the latter case F and G may be of infinite magnitude only at the point at which the datum line cuts the edge line. If in a rectangular coordinate system, with F and G as absciss- respectively ordinate-axes, each pair of values of F and G is represented by a point, the F- and G-curves valid for the datum line will correspond to an ellipse or circle according to equation (7.12). The latter will occur when

$$v_x'^2 + v_z'^2 = u_x'^2 + u_z'^2, \text{ and } v_x'u_x' + v_z'u_z' = 0$$

If the mapped values of F and G that form the basis of the construction of the "relation figure" differ with constant values ΔF and ΔG from the real anomaly values of these components, we still obtain the same "relation figure" but its centre is displaced. If the real values are $F - \Delta F$ and $G - \Delta G$ the centre is removed at a distance ΔF in the direction of the F-axis and at distance ΔG in the direction of the G-axis.

§ 8. Calculations from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$.

In the following we make use of formulae given in § 3.

As T and the angles i, ϑ, α and γ are given, we may also consider T' and the angles i' and v_i as known, since the latter quantities can be calculated with the aid of the equations in (3.2). The case is analogous as regards I'_r, k', ψ', v_r, when I_r, k, ψ and ϑ_r are given and concerning φ', when φ is given.

The calculations are based on the formulae (3.1), which we write

$$\left.\begin{aligned} \varepsilon \varkappa T'' \cos (i'' - \varphi') &= \varepsilon M_{//}(1 + \varkappa N_{//}), \\ \varepsilon \varkappa T'' \sin (i'' - \varphi') &= \varepsilon M_{\perp}(1 + \varkappa N_{\perp}). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.1)$$

At first we treat the simple case, when $4\pi\varkappa \ll 1$.

A. $4\pi\varkappa \ll 1$.

As in this case the quantities $\varkappa N_{//}$ and $\varkappa N_{\perp}$ can be omitted, we obtain from (8.1) the formulae

$$\left.\begin{aligned} \tan (i'' - \varphi') &= \frac{\varepsilon M_{\perp}}{\varepsilon M_{//}}, \\ \varepsilon\varkappa T'' &= \sqrt{(\varepsilon M_{//})^2 + (\varepsilon M_{\perp})^2} = \varepsilon M_T. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8.2)$$

With the aid of the above formulae it is possible to calculate φ and ε, when we know the magnetic susceptibility ($\varkappa$) and the natural remanence of the sheet material. If only k' and ψ' are given, it is possible to calculate φ and the product $\varepsilon\varkappa$.

We separate the following two cases:

a) The natural remanence has the same direction as the earth's normal field, i. e. $i'' = i'$ and $k' = k$.

From the value of $(i' - \varphi')$, obtained from the first expression in (8.2), it is possible to determine the angle φ with the aid of the relation

$$\cos \varphi = \cos \varphi' \ \cos \gamma.$$

Further the second equation in (8.2) gives

$$\varepsilon\varkappa\,(1 + k) = \frac{\varepsilon M_T}{T'}.$$

b) The dip angle (φ) and the product ($\varepsilon\varkappa$) are known.

As the angle φ' can be considered as known, the equations (8.2) give the values of i'' and T''. From these quantities we arrive to k' and ψ' with the aid of the formulae

$$\left.\begin{aligned} k' &= \sqrt{1 - 2\,\frac{T''}{T'}\cos\,(i''-i') + \left(\frac{T''}{T'}\right)^2}, \\ \tan\,\psi' &= \frac{T''\,\sin\,i'' - T'\,\sin\,i'}{T''\,\cos\,i'' - T'\,\cos\,i'}. \end{aligned}\right\} \qquad (8.3)$$

B. *$\varkappa N_{//}$ and $\varkappa N_{\perp}$ can not be omitted.*

The two equations in (8.1) contain three independent variables, viz. $\varkappa$, the product $\varepsilon T''$ and the angle $(i''-\varphi')$. If one of these variables is known, the other two can be calculated with the aid of the following formulae:

1) *$\varkappa$ known.*

$$\left.\begin{aligned} \tan\,(i''-\varphi') &= \frac{\varepsilon M_{\perp}\,(1+\varkappa N_{\perp})}{\varepsilon M_{//}\,(1+\varkappa N_{//})}, \\ \varepsilon T'' &= \frac{\sqrt{(\varepsilon M_{//})^2\,(1+\varkappa N_{//})^2 + (\varepsilon M_{\perp})^2\,(1+\varkappa N_{\perp})^2}}{\varkappa}. \end{aligned}\right\} \qquad (8.4)$$

2) *$\varepsilon T''$ known.*

$$\left.\begin{aligned} \varkappa &= \frac{N_{//}(\varepsilon M_{//})^2 + N_{\perp}(\varepsilon M_{\perp})^2 + \sqrt{(\varepsilon T'')^2\,(\varepsilon M_{T})^2 - (\varepsilon M_{//}\varepsilon M_{\perp})^2\,(N_{\perp}-N_{//})^2}}{(\varepsilon T'')^2 - (N_{//}\,\varepsilon M_{//})^2 - (N_{\perp}\,\varepsilon M_{\perp})^2}, \\ \tan\,(i''-\varphi') &= \frac{\varepsilon M_{\perp}\,(1+\varkappa N_{\perp})}{\varepsilon M_{//}\,(1+\varkappa N_{//})}. \end{aligned}\right\} \qquad (8.5)$$

3) *The angle $(i''-\varphi')$ known.*

$$\left.\begin{aligned} \varkappa &= \frac{\varepsilon M_{\perp} - \varepsilon M_{//}\,\tan\,(i''-\varphi')}{N_{//}\,\varepsilon M_{//}\,\tan\,(i''-\varphi') - N_{\perp}\,\varepsilon M_{\perp}} \\ &= \frac{\varepsilon M_{//}\sin\,(i''-\varphi') - \varepsilon M_{\perp}\,\cos\,(i''-\varphi')}{N_{\perp}\,\varepsilon M_{\perp}\,\cos\,(i''-\varphi') - N_{//}\,\varepsilon M_{//}\,\sin\,(i''-\varphi')}, \\ \varepsilon\varkappa T'' &= \frac{\varepsilon M_{//}\,\varepsilon M_{\perp}\,(N_{\perp}-N_{//})}{N_{\perp}\,\varepsilon M_{\perp}\,\cos\,(i''-\varphi') - N_{//}\,\varepsilon M_{//}\,\sin\,(i''-\varphi')} \end{aligned}\right\} \qquad (8.6)$$

We separate the following two cases:

a) The natural remanence has the same direction as the earth's normal field, i. e. $i'' = i'$ and $k' = k$.

When $\varkappa$ is known, we obtain φ and $\varepsilon(1+k)$ with the aid of the formulae (8.4).

When $\varepsilon T'' = \varepsilon T'(1+k)$ is known, $\varkappa$ and φ can be calculated from the equations (8.5).

When φ is known, $\varkappa$ and $\varepsilon(1 + k)$ are obtained from (8.6).

b) The dip angle (φ) and the quantities ($\varkappa$) and (ε) are given.

From the formulae (8.4) we obtain the values of T'' and i''. After that we can calculate k' and the angle ψ' with the aid of the expressions (8.3).

§ 9. Characteristic properties of the anomaly field.

From the above (§§ 6—7) we have seen that the interpretation equation (6.1) may be regarded as the general equation for the anomaly F along an arbitrarily chosen straight line. On the other hand, if the interpretation equation is to refer to a real sheet the parameter functions must be real and in addition it is necessary that the expression $-(4b_0 + b_1^2)$ is ≥ 0. From these premises we deduce in the following some general characteristic properties of the anomalous field along a straight line.

Extreme Points.

According to the interpretation equation (6.1) the x-coordinates of the extreme points are governed by the condition

$$\frac{dF}{dx} = \frac{-a_1 x^2 - 2\,a_0 x - a_1 b_0 + a_0 b_1}{(x^2 - b_1 x - b_0)^2} = 0$$

If the denominator in this expression is zero $F = \infty$. Apart from this special case, and if the coordinates of the extreme points are indicated by the suffix (e) we obtain

for $a_1 \neq 0$

$$\left.\begin{aligned} x_e &= \frac{-a_0 \pm \sqrt{a_0^2 + a_0 a_1 b_1 - a_1^2 b_0}}{a_1}, \\ F_e &= \frac{a_1}{2\,x_e - b_1} = \frac{a_1^2}{-2\,a_0 - a_1 b_1 \pm 2\sqrt{a_0^2 + a_0 a_1 b_1 - a_1^2 b_0}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ (9.1)$$

and for $a_1 = 0$

$$\left.\begin{aligned} x_e &= \frac{b_1}{2}, \\ F_e &= -\frac{a_0}{x_e^2 + b_0} = -\frac{4\,a_0}{4\,b_0 + b_1^2}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ (9.2)$$

When $a_1 \neq 0$, one extreme value (maximum) of F is always positive while the other extreme (minimum) is always negative. When $a_1 = 0$, the extreme is a maximum if a_0 is positive and a minimum if a_0 is negative.

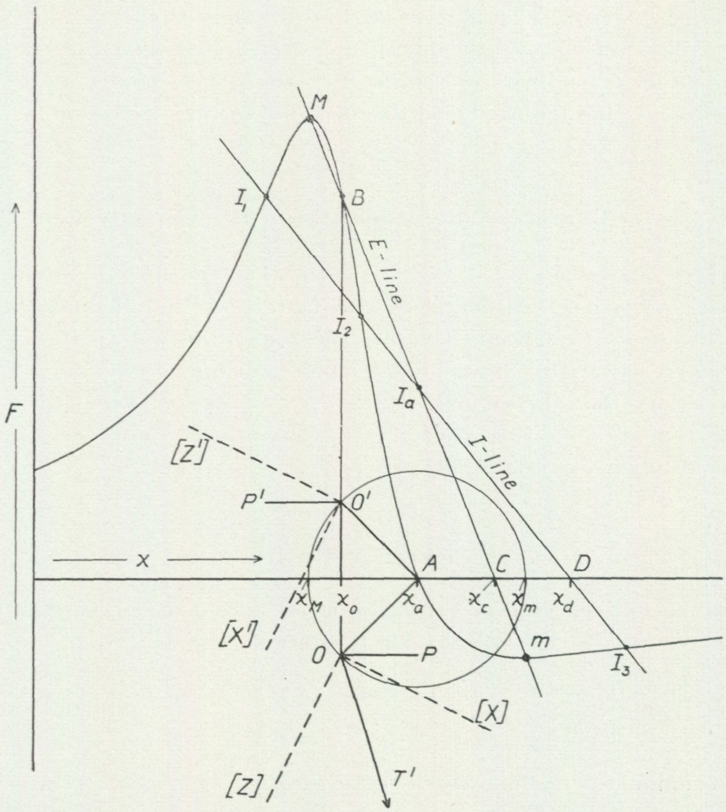

Fig. 9. 1. Positions of the extreme points (M and m) and the points of inflexion (I_1, I_2, I_3) on a magnetic profile curve of F.

Possible positions (O — $[X]$, O' — $[X']$ and O — $[Z]$, O' — $[Z']$) of the sheet if $F = X$ or $F = Z$ and the magnetization of the sheet has the same direction as T'.

According to the interpretation equation the abscissa (x_a) of the point (A) on the profile line where $F = 0$ is

$$x_a = -\frac{a_0}{a_1}.$$

Thus it is apparent from the first equation in (*9.1*) that the point (A) is situated midway between the points of maximum and minimum values of F. When $a_1 = 0$, F does not assume the value zero at any point but approaches this value asymptotically as $x \rightarrow \infty$.

Consider a straight line (E) joining the two extreme points on a curve in a diagram, as in Fig. *9.1*. If we designate the coordinates of the maximum and minimum points by suffix (M) and (m) respectively, the equation for the line E may be written

$$\frac{F - F_M}{x - x_M} = \frac{F - F_m}{x - x_m}.$$

On inserting the expressions for the coordinates of the extreme points, obtained from (9.1), in the above equation, it may be converted to the form

$$F = -\frac{2\, a_1 x + 4\, a_0 + a_1 b_1}{4\, b_0 + b_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.3)$$

This line intersects the curve of F at the extreme points as well as at yet another point (B). The coordinates of this third point of intersection are

$$\left.\begin{aligned} x_b &= \frac{b_1}{2}, \\ F_b &= F_M + F_m = -\frac{4\, a_0 + 2\, a_1 b_1}{4\, b_0 + b_1^2} \\ &= 2\frac{\varepsilon M_{/\!/} (v_z' \cos\alpha + p\, v_x' \cos\gamma) - \varepsilon M_{\perp} (v_x' \cos\alpha - p\, v_z' \cos\gamma)}{t_0 \cos\alpha\ \cos\gamma}. \end{aligned}\right\} \quad \ldots\ldots \quad (9.4)$$

The line E also intersects the x-axis in a point (C) whose coordinate is

$$\left.\begin{aligned} x_c &= -2\frac{a_0}{a_1} - \frac{b_1}{2} = 2\, x_a - x_b, \\ \text{or} \qquad x_c - x_a &= x_a - x_b. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (9.5)$$

This obviously implies that the points x_b and x_c on the x-axis in Fig. 9.1 are situated at the same distance from, but on opposite sides of the point (A), where $F = 0$.

When $\alpha = \beta = \gamma = 0$ we obtain

$$\left.\begin{aligned} x_b &= x_0, \\ F_b &= F_M + F_m = 2\frac{\varepsilon M_{/\!/} v_z - \varepsilon M_{\perp} v_x}{t_0}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (9.6)$$

The total amplitude of the F anomaly along a straight line is

$$\left.\begin{aligned} F_M - F_m &= -\frac{2\, a_1 (x_M - x_m)}{(2\, x_M - b_1)(2\, x_m - b_1)} = -\frac{4\sqrt{a_0^2 + a_0 a_1 b_1 - a_1^2 b_0}}{4\, b_0 + b_1^2} \\ &= \frac{2\,\varepsilon M_T}{r_{\perp}} \sin v, \end{aligned}\right\} \quad \ldots\ldots \quad (9.7)$$

where $r_{\perp} = t_0 \dfrac{\sqrt{\cos\alpha\ \cos\gamma}}{\mu^2}$ is the shortest distance between the profile line and the edge of the sheet and $\sin v = \sqrt{v'^2_x + v'^2_z}$, where v is the angle between the direction of F and the edge of the sheet.

Thus, for a given sheet, the total amplitude of the anomaly (F) along a certain arbitrary straight line is equal to the double product of the thickness (ε), the total moment (M_T) per unit volume and the sine of the angle (v) between F and the edge, divided by the distance ($r_{\perp}$) between the profile line and the edge.

Points of Inflexion.

The condition $\frac{d^2 F}{d x^2} = 0$, applied to the interpretation equation, gives the following equation for determining the abscissae of the points of inflexion (x_I).

$$a_1 x^3 + 3 a_0 x^2 + 3 (a_1 b_0 - a_0 b_1) x = a_1 b_0 b_1 - a_0 (b_0 - b_1^2). \quad \ldots\ldots\ldots\ldots \quad (9.8)$$

For $a_1 = 0$ we obtain the values of

$$\left.\begin{aligned} x_I &= \tfrac{1}{2}\left[b_1 \pm \sqrt{\frac{4 b_0 + b_1^2}{3}}\right] = x_e \pm \tfrac{1}{2}\sqrt{-\frac{4 b_0 + b_1^2}{3}}, \\ &\text{Further for both values of } x_I \text{ we obtain } F_I = -\frac{3 a_0}{4 b_0 + b_1^2} = \tfrac{3}{4} F_e. \end{aligned}\right\} \quad \ldots\ldots \quad (9.9)$$

In this case, there are thus two points of inflexion, situated symmetrically in relation to the extreme point and having the same value of F.

If $a_1 \neq 0$ there are always three points of inflexion. To illustrate this move the zero point on the x-axis in Fig. 9.1. to point A. This implies that $a_0 = 0$ and that equation (9.8) assumes the form

$$x^3 + 3 b_0 x = b_0 b_1.$$

This equation has three real roots when $b_0^3 + \frac{b_0^2 b_1^2}{4} = \frac{b_0^2 (4 b_0 + b_1^2)}{4} < 0.$

This condition is always fulfilled since $(4 b_0 + b_1^2)$ must be negative in order that the magnetic parameters of the sheet be real.

It is also possible to show that the three points of inflexion are always situated on a straight line (I) in a diagram according to Fig. 9.1. If we put $F = Ax + B$ in the interpretation equation, it is possible to determine A and B so that the mentioned equation becomes identical with equation (9.8). For the I-line we obtain the equation

$$F = -\frac{a_1 x + 3 a_0 + a_1 b_1}{4 b_0 + b_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.10)$$

This line intersects the x-axis at the point (D), where

$$x_d = -\frac{3 a_0 + a_1 b_1}{a_1} = x_a + 2 (x_c - x_a) \quad \ldots\ldots\ldots\ldots \quad (9.11)$$

and the E-line (equ. 9.3) at the point I_a. This point has the coordinates

$$\left.\begin{aligned} x &= -\frac{a_0}{a_1} = x_a, \\ F &= \frac{2 a_0 + a_1 b_1}{-(4 b_0 + b_1^2)} = \tfrac{1}{2} F_b. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (9.12)$$

Thus the I-line in Fig. 9.1 passes through the point of intersection between the E-line and a line through point A, parallel to the F-axis. It cuts the x-axis on the same side of A as does the E-line, but at a distance from A twice that to the point C.

It is hence possible to construct the line on which the points of inflexion are situated, knowing only the positions of the extreme points.

§ 10. Graphical interpretation.

We are only investigating the case in which the edge of the sheet is horizontal, and in which the F-curve may be referred to an orthogonal horizontal profile line, i. e. $\alpha = \beta = \gamma = 0$. The following is to be taken in conjunction with the description of the characteristic properties of the anomalous field (in § 9). x_b, however, is to be replaced by x_0, since these quantities are identical on account of the above assumptions for α, β and γ. Further, the notations for the abscissae are also used as designations of the corresponding points on the x-axis.

In the following we treat the methods of graphical interpretation for a number of special cases.

a) GIVEN THE POSITION OF THE TWO EXTREME POINTS (M AND m).

As has been shown earlier we have

$$x_0 = \tfrac{1}{2} b_1, \; x_a = -\frac{a_0}{a_1}.$$

If we take the point x_a (point A in fig. *9.1*) as origo on the x-axis, $a_0 = 0$ and $x_e^2 = - b_0$ (9.*1*). Thus we may write

$$t_0 = \pm \sqrt{-b_0 - \left(\frac{b_1}{2}\right)^2} = \pm \sqrt{(x_e - x_a)^2 - (x_0 - x_a)^2} \quad \dots\dots\dots\dots\dots \quad (10.1)$$

The point x_a (A in Fig. *9.1*), is situated half-way between x_M and x_m. On drawing the E-line we obtain the position of point x_0 by marking off from A the distance to x_c in the opposite direction.

Further (comp. *10.1*), if we describe a circle (see Fig. *9.1*) with centre in A and the radius $/x_M - x_a/ = /x_m - x_a/$, and then draw a normal to the x-axis through x_0, the edge of the sheet is situated at one of the two points O and O′ at which the normal cuts this circle.

According to (*9.4*) we have

$$2\, a_0 + a_1 b_1 = 2\, t_0^2\, (F_M + F_m) = 2\, t_0^2\, F_b$$

and according to (*9.1*)

$$a_1 = (2\, x_e - b_1)\, F_e = 2\, (x_e - x_0)\, F_e.$$

On inserting these expressions in equations (*7.7*) we obtain

$$\left.\begin{aligned} \varepsilon M_{//} &= \frac{t_0 F_b v_z - 2\,(x_e - x_0)\, F_e v_x}{2\,(v_x^2 + v_z^2)}, \\ \varepsilon M_{\perp} &= -\frac{t_0 F_b v_x + 2\,(x_e - x_0)\, F_e v_z}{2\,(v_x^2 + v_z^2)}, \end{aligned}\right\} \quad \dots\dots\dots\dots\dots \quad (10.2)$$

where x_e and F_e refer to the same extreme value; but immaterial whether it is a maximum or minimum. The quantitites F_e, and $F_b = F_M + F_m$ are given.

The quantities t_0 and $(x_e - x_0)$ obtained from the construction diagram (Fig. *9.1*), are to be given their appropriate signs.

The ratio $\dfrac{M_\perp}{M_{//}} = \dfrac{t_0 v_z - (x_a - x_0) v_x}{t_0 v_x + (x_a - x_0) v_z}.$

When $F = X$; i. e. $v_x = 1$, $v_z = 0$, the ratio

$$\frac{M_\perp}{M_{//}} = - \frac{x_a - x_0}{t_0}, \qquad (10.3)$$

and when $F = Z$, i. e. $v_x = 0$, $v_z = 1$, the ratio

$$\frac{M_\perp}{M_{//}} = \frac{t_0}{x_a - x_0}. \qquad (10.4)$$

Further, if the directions of the permanent and induced magnetism of the sheet coincide and if $4 \pi \varkappa \leqslant 1$ we have according to (*8.2*) that

$$\frac{M_\perp}{M_{//}} = \tan\,(i' - \varphi).$$

The dip of the sheet may hence be estimated from an X- or Z-curve according to the following simple rule. Rotate the angles $x_0 O A (x_0 O' A)$ and $A\,O\,P$ $(A\,O'P')$ in Fig. *9.1* ($O\,P$ and $O'\,P'$ are parallel with the x-axis) round their apices at O (O') until the side $O\,A$ $(O'A)$ coincides with the direction T'. Then the position of the sheet is given by the line Ox_0 $(O'x_0)$ if $F = X$, and by the line $O\,P$ $(O'P')$ if $F = Z$. In this way the dashed lines $[X]$ and $[Z]$ originating in O and the lines $[X']$ and $[Z']$ originating in O' in Fig. *9.1* have been obtained. Thus these lines indicate the two possible positions of the sheet if, by chance, the F-curve reproduced in the diagram should represent the course of the field component X or Z. It must, however, be emphasized that the above holds only on condition that the sheet fulfils the magnetic requirements stated above.

b) Given four points $(x_1 F_1, x_1' F_1, x_2 F_2, x_2' F_2)$, having pairwise a common value of F.

The following notations are used:

$$x_g = \tfrac{1}{2}(x_1 + x_1'), \quad x_g' = \tfrac{1}{2}(x_2 + x_2').$$

The point x_g is hence situated midway between points x_1 and x_1', and x_g' midway between points x_2 and x_2'. In the following, when carrying out the graphical constructions, we assume the points x_g and x_g' to be plotted already in the construction diagram.

According to the formulae (*16.3*) we have

$$\left.\begin{aligned} a_0 &= \frac{(x_1 x_1' - x_2 x_2')\,F_1 F_2}{F_1 - F_2}, & b_0 &= -\frac{x_1 x_1' F_1 - x_2 x_2' F_2}{F_1 - F_2}, \\ a_1 &= -\frac{2\,(x_g - x_g')\,F_1 F_2}{F_1 - F_2}, & b_1 &= \frac{2\,(x_g F_1 - x_g' F_2)}{F_1 - F_2}. \end{aligned}\right\} \qquad (10.5)$$

Thus we obtain from the above expression of b_1

$$\left.\begin{aligned} x_0 - x_g &= \tfrac{1}{2}\, b_1 - x_g = \frac{(x_g - x_g')\, F_2}{F_1 - F_2}, \\ x_0 - x_g' &= \tfrac{1}{2}\, b_1 - x'_g = \frac{(x_g - x_g')\, F_1}{F_1 - F_2}, \end{aligned}\right\} \qquad (10.6)$$

or $$\frac{x_0 - x_g}{x_0 - x_g'} = \frac{F_2}{F_1}. \qquad (10.7)$$

Further

$$x_a = -\frac{a_0}{a_1} = \frac{x_1 x_1' - x_2 x_2'}{2\,(x_g - x_g')},$$

where $(x_g - x_g')$ is invariable with the chose of fiducial mark on the x-axis, as this expression is a distance between two points. Consequently, we may write

$$x_a - x_1 = -\frac{(x_2 - x_1)\,(x_2' - x_1)}{2\,(x_g - x_g')}, \quad x_a - x_2 = \frac{(x_1 - x_2)\,(x_1' - x_2)}{2\,(x_g - x_g')},$$

$$x_a - x_1' = -\frac{(x_2 - x_1')\,(x_2' - x_1')}{2\,(x_g - x_g')}, \quad x_a - x_2' = \frac{(x_1 - x_2')\,(x_1' - x_2')}{2\,(x_g - x_g')}.$$

Hence

$$\left.\begin{aligned} \frac{x_a - x_1}{x_a - x_2} &= \frac{x_2' - x_1}{x_1' - x_2}, \quad \frac{x_a - x_1'}{x_a - x_2} = \frac{x_2' - x_1'}{x_1 - x_2}, \\ \frac{x_a - x_1}{x_a - x_2'} &= \frac{x_2 - x_1}{x_1' - x_2'}, \quad \frac{x_a - x_1'}{x_a - x_2'} = \frac{x_2 - x_1'}{x_1 - x_2'}. \end{aligned}\right\} \qquad (10.8)$$

If the origin on the x-axis is displaced to x_a, we have $a_0 = 0$ and $x_e^2 = -b_0$ (comp. 9.1). Thus, according to (10.5),

$$(x_1 - x_a)(x_1' - x_a) - (x_2 - x_a)(x_2' - x_a) = 0$$

and

$$-b_0 = (x_e - x_a)^2 = (x_1 - x_a)(x_1' - x_a) = (x_2 - x_a)(x_2' - x_a) \qquad (10.9)$$

From (10.5) and (10.6) we obtain

$$a_1 = -2(x_0 - x_g)F_1 = -2(x_0 - x_g')F_2.$$

Hence

$$F_e = \frac{a_1}{2\,x_e - b_1} = \frac{x_0 - x_g}{x_0 - x_e}\, F_1 = \frac{x_0 - x_g'}{x_0 - x_e}\, F_2. \qquad (10.10)$$

The formulae (10.7—10.9) allow the determination of the positions of the points x_0, x^a and x_e by simple graphical constructions. After that the position of the edge and eventually the dip angle of the sheet, may be determined in the same way as given earlier in the case a).

The quantities F_e and $F_b = F_M + F_m$ may be graphically constructed according to the formula (10.10). It is then possible to calculate $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ from the expression (10.2).

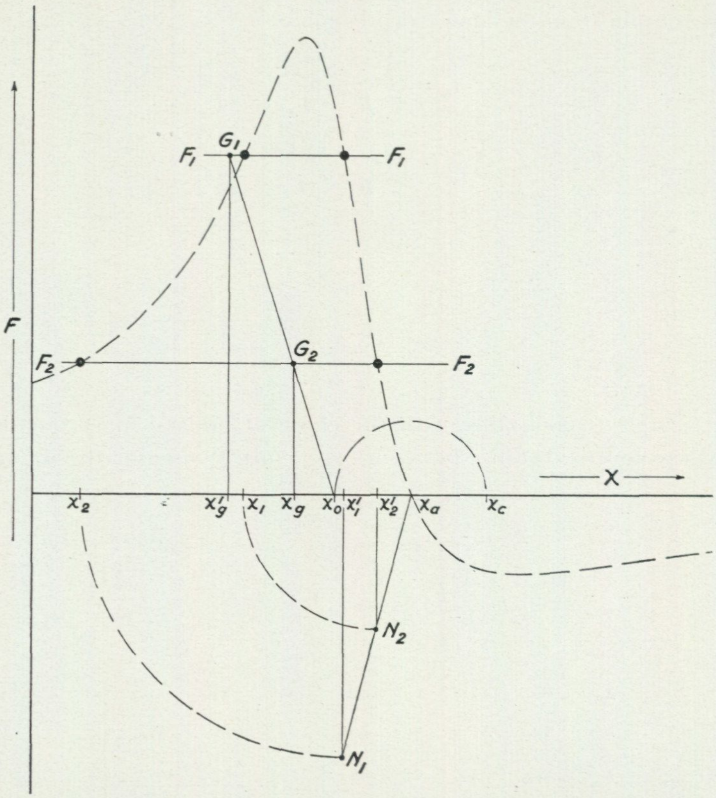

Fig. 10.1. Determination of the points x_0, x_a and x_c by graphical constructions based on four given points (marked with full circles) on a profile curve, viz. two points hawing $F = F_1$ and two points having $F = F_2$, where F_1 and F_2 have the same signs.

The graphical constructions are made as follows:

We first determine the position of x_0, x_a and x_c. This is illustrated in *Fig. 10.1* for the case of F_1 and F_2 having the same signs, and in *Fig. 10.2* for the case of F_1 and F_2 having the opposite signs.

Determination of x_a. Plot point (G_1) having the coordinates x_g' and F_1, and the point (G_2) having the coordinates x_g and F_2. Draw a straight line through (G_1) and (G_2). From equation (*10.7*) it is apparent that this line will intersect the x-axis at the required point x_0.

Determination of x_a. It is sufficient to use one of the four equations given in (*10.8*). The constructions in Fig. 10.1 and Fig. 10.2 are made according to the fourth equation. Thereby the points N_1 and N_2 have been marked off on the normals through x_1' respectively x_2'. The distance from x_1' to N_1 is equal the distance between x_1' and x_2, and the distance from x_2' to N_2 is equal the distance between x_2' and x_1. The straight line through the points N_1 and N_2 cuts the x-axis in the required point x_a.

The quantities $(x_2 - x_1')$ and $(x_1 - x_2')$ in the fourth equation (as the corresponding quantities in the other three equations) under (*10.8*) have always

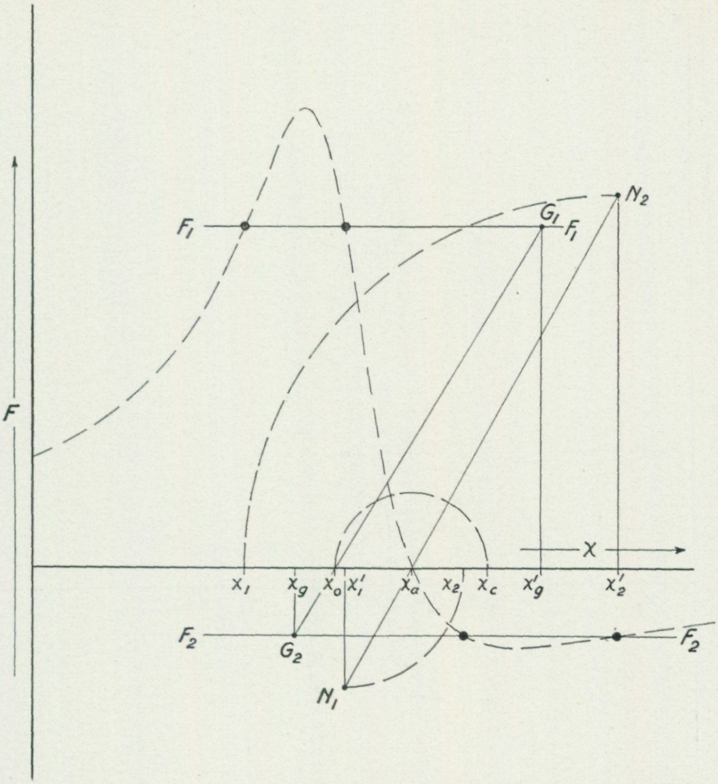

Fig. 10.2. Determination of the points x_0, and x_a and x_c by graphical constructions based on four given points (marked with full circles) on a profile curve, viz. two points having $F = F_1$ and two points having $F = F_2$, where F_1 and F_2 have *opposite* signs.

the same signs, if the signs of F_1 and F_2 are equal, and the opposite signs if the signs of F_1 and F_2 are opposite. Thus the points N_1 and N_2 shall be plotted to the same side of the x-axis in the former case (Fig. 10.1), and to opposite sides of the x-axis in the latter case (Fig. 10.2).

It is of interest to state that without x_a, only the x-coordinates of the datum points enter the formulae (*10.8*). This fact implies that we should obtain the same position of x_a by the graphical construction, even if the *given* anomaly values were $(F_1 + \Delta F)$ and $(F_2 + \Delta F)$, where F_1 and F_2 are the *real* anomaly values and ΔF is a constant quantity. The F-curve in Fig. 10.1, for example, should in the mentioned case be displaced a distance ΔF upward or downward in relation to the x-axis, but the construction lines for x_a should be the same. On the other hand the constructed position of x_0 as well as the cutting point between the F-curve and the x-axis should be changed.

The facts mentioned above may be used to determine the quantity ΔF in the case where uncertainity exists as to the *real* zero-level of a measured profile curve of F. Thereby we may first construct the positions of x_0 and x_a in the way given above, and after that draw a normal to the x-axis through

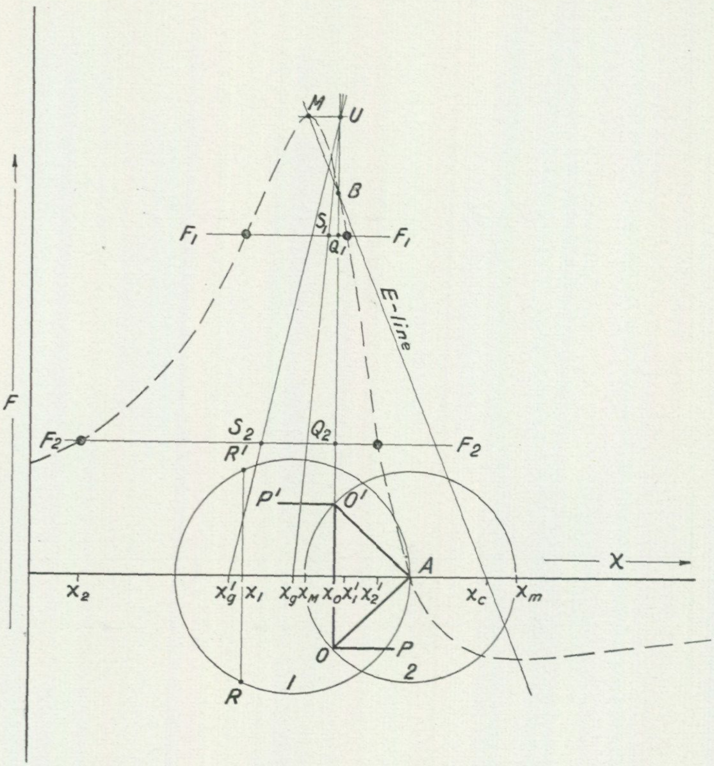

Fig. 10.3. Continuation of the graphical interpretation from Fig. *10.1*. Determination of the points x_M, x_m, O, O', M and B.

x_a. This normal cuts the F-curve on a distance ΔF from x_a. On displacing the x-axis in the construction diagram to a parallel line through the cutting point mentioned, this new x-axis gives the right zero-level of the F-curve and cuts the line through the points G_1 and G_2 in the real head point x_0 and the F-curve in x_a.

Determination of x_c is made by marking off from x_a the distance to x_0 in the opposite direction (comp. § 9).

The graphical constructions described below are illustrated in *Fig. 10.3*, where the F-curve and the given datum points are the same as in Fig. 10.1.

Determination of x_M *and* x_m. Describe a circle (1) with centre x_g (or x_g') passing through point x_a (denoted A in Fig. 10.3), and draw a normal to the x-axis through x_1 (resp. x_2) or x_1' (resp. x_2'). This normal cuts circle (1) in two points (R and R') and the distance between point R or R' and the x-axis is, according to equation (*10.9*), the same as the distance between x_e and x_a. On drawing a circle (2) with center in A and a radius equal to the mentioned distance, this circle cuts the x-axis in x_M and x_m.

Determination of the position of the edge. On drawing a normal to the x-axis through x_0, the edge of the sheet is situated at one of the two points (O and O'), where the mentioned normal cuts circle (2).

Determination of the points M and B. The maximum point (M) of the F-curve may be determined with the aid of one of the two points

S_1 with the coordinates: $x = x_0 + (x_g - x_M)$, $F = F_1$,
and S_2 » » » : $x = x_0 + (x_g' - x_M)$, $F = F_2$.

We obtain the position of S_1 by marking off the distance $(x_g - x_M)$ at the line $F = F_1$ from the point of intersection (Q_1) of this line and the normal through x_0. S_2 is situated at the line $F = F_2$ at a distance $(x_g' - x_M)$ from the corresponding point of intersection (Q_2).

By drawing a straight line through x_g and S_1, and another line through x_g' and S_2, we find that both these lines cut the previously mentioned normal from x_0 at a point U. The F-coordinate of U is according to (*10.10*) equal to F_M. As x_M is known we can now plot the point M in the construction diagram.

As the points M and x_c are known it is possible to draw the E-line. The point B which indicates F_b is identical with the point of intersection between this line and the normal from x_0.

SPECIAL CASE. GIVEN AN EXTREME POINT (M OR m) AND ITS CORRESPONDING POINTS (H AND H') OF HALF-VALUE.

We have: $F_1 = 2F_2 = F_e$, $x_1 = x_1' = x_e$, $x_2 = x_h$, $x_2' = x_h'$, $x_g = x_g$, $x_g' = {}^1/_2\,(x_h + x_h')$.

The equations (*10.7*) and (*10.8*) may hence be written:

$$\left.\begin{aligned} & x_0 - x_e = -\,(x_g' - x_e), \\ & \frac{x_a - x_e}{x_a - x_h} = \frac{x_h' - x_e}{x_e - x_h}, \quad \frac{x_a - x_e}{x_a - x_h'} = \frac{x_h - x_e}{x_e - x_h'}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (10.11)$$

After the determination of the points x_0 and x_a according to the equations above we obtain immediately point x_c by marking off from x_a the distance to x_0 in the opposite direction on the x-axis. The remaining graphical constructions are identical with those in the case a).

The well-known thumb rule that for a Z-curve $t_0 = \pm\,{}^1/_2(x_h - x_h')$ includes an approximation. The error caused by this approximation is illustrated in *Fig. 10.4.* There the graphical constructions are based on the first equation in (*10.11*) and the formulae

$$\left.\begin{aligned} & t_0 = \pm\sqrt{-\,b_0 - \left(\frac{b_1}{2}\right)^2} = \pm\sqrt{-\,(x_h - x_e)(x_h' - x_e) - (x_0 - x_e)^2}, \\ & x_k - x_g' = -\,(x_e - x_g'), \\ & x_a - x_e = -\,\frac{(x_h - x_e)\,(x_h' - x_e)}{x_e - x_k}. \end{aligned}\right\} \quad \ldots \quad (10.12)$$

The first formula above is obtained by using the expressions of b_0 and b_1 from (*10.5*) and choosing x_e as fiducial mark. The second formula defines the point x_k which is used by determining point x_a according to the third formula.

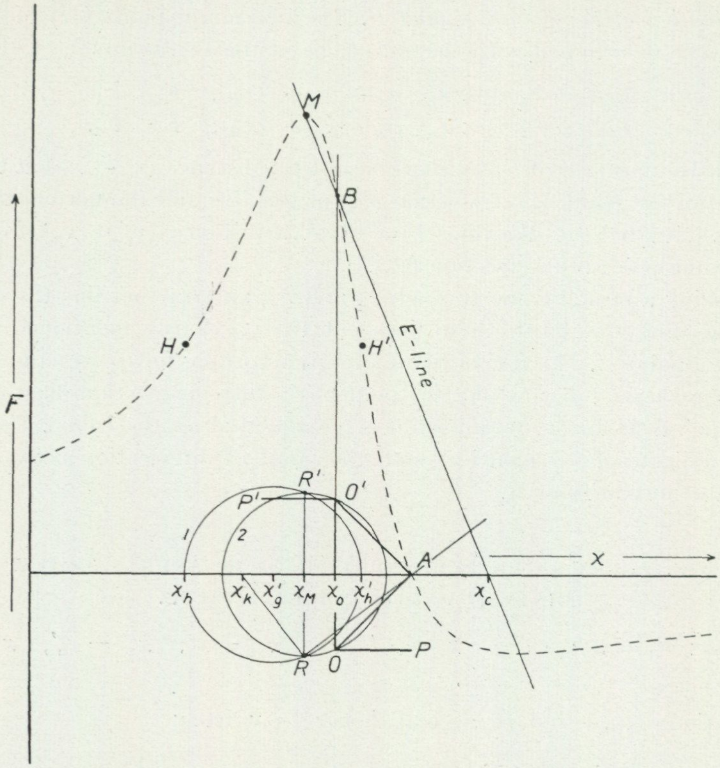

Fig. 10.4. Constructions by graphical interpretation based on three given points on a profile curve, viz. one extreme point (M) and its corresponding points (H and H') of half-value.

The graphical constructions may be performed as follows. From points x_e (x_M in Fig. 10.4) and x_g' mark off the distance between these two points in opposite directions on the x-axis, from where we obtain the positions of x_0 and x_k respectively. Draw normals to the x-axis through x_e and x_0 and describe a circle (1) with center x_g', passing through x_h and x_h'. Further, draw another circle (2) with center x_e, passing through the points of intersection (R and R') of circle (1) with the normal from x_e. Then the edge of the sheet is situated at one of the points (O and O'), where the normal through x_0 cuts the circle (2).

Draw a straight line through the points x_k and R (or R'). The normal to this line through R (resp. R') intersects the x-axis at x_a (A in Fig. 10.4).

According to the thumb rule mentioned above, t_0 should be equal to the distance between x_g' and x_h in Fig. 10.4. The real value of t_0, however, is equal to the distance between x_0 and O. The graphical constructions show that the latter distance is always less than the former, if x_g' does not coincide with x_e. If $x_g' = x_e$, also $x_0 = x_e$, so the thumb rule in this case is exact. These facts are obviously valid whether the profile curve corresponds to a component Z or an arbitrary component F.

For the ratio between t_0 and $(x_h - x_g')$ one may derive the formula

$$\frac{t_0}{x_h - x_g'} = \pm\sqrt{1 - 2\left(\frac{x_e - x_g'}{x_h - x_g'}\right)^2} \quad \ldots\ldots\ldots\ldots \quad (10.13)$$

The well-known rule that for an X-curve $t_0 = \pm (x_e - x_a)$ is obviously exact only if $x_0 = x_a$ (comp. Fig. 9.1. and 10.4). In other cases the distance to the edge is always less than the distance between x_e and x_a. For the ratio between these two distances the following formula is valid, viz.

$$\frac{t_0}{x_e - x_a} = \pm\sqrt{1 - \left(\frac{x_0 - x_a}{x_e - x_a}\right)^2} \quad \ldots\ldots\ldots\ldots \quad (10.14)$$

CHAPTER III.

Methods of Interpretation. Single Sheet of Finite Length.

§ 11. Modifying of fundamental formulae.

The formulae (*2.8*) for X and Z at a sheet of finite length are too complicated to be used by a direct calculation of the magnetic parameters of a sheet. It is possible, however, to give these formulae more suitable forms by expanding in series.

Such a modification of the basic formulae involves a limitation as regards their range of validity. In the following, simplified formulae are derived for two different conditions, viz. $x^2 < l^2 - t^2$ and $x^2 < t^2$.

A. $x^2 < l^2 - t^2$.

Putting

$$f(x) = \frac{1}{\sqrt{x^2 + t^2 + l^2}\left[t \sin \varphi + x \cos \varphi + \sqrt{x^2 + t^2 + l^2}\right]},$$

$$\Phi(x^2 + t^2) = \frac{1}{x^2 + t^2}\left[\frac{l}{\sqrt{x^2 + t^2 + l^2}} - 1\right],$$

the equations (*2.8*) can be written

$$X_l = -2\,\frac{\varepsilon M_{/\!/} x + \varepsilon M_{\perp} t}{x^2 + t^2} - 2\,(\varepsilon M_{/\!/} x + \varepsilon M_{\perp} t)\,\Phi(x^2 + t^2) + 2\,\varepsilon M_{\perp} l \sin \varphi \cdot f(x).$$

$$Z_l = 2\,\frac{\varepsilon M_{/\!/} t - \varepsilon M_{\perp} x}{x^2 + t^2} + 2\,(\varepsilon M_{/\!/} t - \varepsilon M_{\perp} x)\,\Phi(x^2 + t^2) + 2\,\varepsilon M_{\perp} l \cos \varphi \cdot f(x).$$

Expanding $f(x)$ in series according to Maclaurin's formula we obtain $f(x) = n_0 + n_1 x + n_2 x^2 + n_3 x^3 + \ldots$, where

$$\left.\begin{aligned} n_0 &= f(0) = \frac{1}{\sqrt{t^2 + l^2}\left[t \sin \varphi + \sqrt{t^2 + l^2}\right]}, \\ n_1 &= \frac{f'(0)}{1} = -\,n_0 \frac{\cos \varphi}{t \sin \varphi + \sqrt{t^2 + l^2}}, \\ n_2 &= \frac{f''(0)}{1 \cdot 2} = -\frac{n_0}{2}\left[n_0 - 2\left(\frac{n_1}{n_0}\right)^2 + \frac{1}{t^2 + l^2}\right], \\ n_3 &= \frac{f'''(0)}{1 \cdot 2 \cdot 3} = -\frac{n_1}{2}\left[n_0 - \frac{2\,n_2}{n_0}\right]. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11.1)$$

The expansion is valid if $x^2 < t^2 + l^2$.

In the same manner we obtain

$\Phi(x^2 + t^2) = m_0 + m_1 x^2 + m_2 x^4 + m_3 x^6 + \ldots$, where

$$\left.\begin{aligned}
m_0 &= \Phi(t^2) = \frac{1}{t^2}\left[\frac{l}{\sqrt{t^2 + l^2}} - 1\right],\\
m_1 &= \frac{\Phi'(t^2)}{1} = -\frac{1}{t^2}\left[m_0 + \frac{l}{2\ (t^2 + l^2)^{3/2}}\right],\\
m_2 &= \frac{\Phi''(t^2)}{1\cdot 2} = -\frac{1}{t^2}\left[2\ m_1 - \frac{3\,l}{4\ (t^2 + l^2)^{5/2}}\right],\\
m_3 &= \frac{\Phi'''(t^2)}{1\cdot 2\cdot 3} = -\frac{1}{t^2}\left[\frac{3\ m_2}{2} + \frac{15\,l}{8\ (t^2 + l^2)^{7/2}}\right],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
m_n &= \frac{\Phi^{(n)}(t^2)}{n!} = -\frac{1}{t^2}\left[\frac{n\cdot m_{(n-1)}}{(n-1)!} + (-1)^{(n+1)}\frac{(2\,n-1)!!\,l}{2^n\,(t^2 + l^2)^{(2\,n+1)/2}}\right].
\end{aligned}\right\} \quad \ldots\ldots \quad (II.2)$$

The expansion is valid for $x^2 < l^2 - t^2$.

Using the expressions (*II.1*) and (*II.2*), we can write the formulae above for X_l and Z_l in the forms

$$\left.\begin{aligned}
&X_l = -2\,\frac{\varepsilon M_{//}x + \varepsilon M_\perp t}{x^2 + t^2} + h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \ldots\\
&\text{where}\\
&\qquad h_0 = 2\ \varepsilon M_\perp(-\ m_0 t + n_0 l\ \sin\ \varphi),\\
&\qquad h_1 = 2\ (-\ m_0\varepsilon M_{//} + n_1 l\,\varepsilon M_\perp\ \sin\ \varphi),\\
&\qquad h_2 = 2\ \varepsilon M_\perp(-\ m_1 t + n_2 l\ \sin\ \varphi),\\
&\qquad h_3 = 2\ (-\ m_1\varepsilon M_{//} + n_3 l\,\varepsilon M_\perp\ \sin\ \varphi),\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (II.3)$$

and

$$\left.\begin{aligned}
&Z_l = 2\,\frac{\varepsilon M_{//}t - \varepsilon M_\perp x}{x^2 + t^2} + k_0 + k_1 x_1 + k_2 x^2 + k_3 x^3 + \ldots\\
&\text{where}\\
&\qquad k_0 = 2\ (m_0 t\,\varepsilon M_{//} + n_0 l\,\varepsilon M_\perp\ \cos\ \varphi),\\
&\qquad k_1 = 2\ \varepsilon M_\perp(-\ m_0 + n_1 l\ \cos\ \varphi),\\
&\qquad k_2 = 2\ (m_1 t\,\varepsilon M_{//} + n_2 l\,\varepsilon\,M_\perp\ \cos\ \varphi),\\
&\qquad k_3 = 2\ \varepsilon M_\perp(-\ m_1 + n_3 l\ \cos\ \varphi),\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \ldots\ldots \quad (II.4)$$

B. $x^2 < t^2$.

Putting

$$\Phi_B(x^2) = \frac{1}{x^2 + t^2}\left[\frac{1}{\sqrt{x^2 + t^2 + l^2}} - \frac{1}{\sqrt{t^2 + l^2}}\right],$$

equations (2.8) can be written

$$X_l = -2\frac{l}{\sqrt{t^2+l^2}}\cdot\frac{\varepsilon M_{/\!/}x+\varepsilon M_{\perp}t}{x^2+t^2} - $$
$$-2\,(\varepsilon M_{/\!/}x+\varepsilon M_{\perp}t)\;l\cdot\Phi_B(x^2)+2\,\varepsilon M_{\perp}\,l\sin\varphi\cdot f(x),$$

$$Z_l = 2\frac{l}{\sqrt{t^2+l^2}}\cdot\frac{\varepsilon M_{/\!/}t-\varepsilon M_{\perp}x}{x^2+t^2} + $$
$$+\,2\,(\varepsilon M_{/\!/}t-\varepsilon M_{\perp}x)\;l\cdot\Phi_B(x^2)+2\,\varepsilon M_{\perp}\,l\cos\varphi\cdot f(x).$$

Expanding $\Phi_B(x^2)$ in series in regard to x^2, we obtain

$$\left.\begin{aligned}
&\Phi_B(x^2) = m_0 + m_1x^2 + m_2x^4 + m_3x^6 + \ldots,\ \text{where}\\
&m_0 = \Phi_B(0) = 0\\
&m_1 = \frac{\Phi_B'(0)}{1} = -\frac{1}{t^2}\cdot\frac{1}{2\,(t^2+l^2)^{3/2}},\\
&m_2 = \frac{\Phi_B''(0)}{1\cdot 2} = -\frac{1}{t^2}\left[m_1 - \frac{3}{8\,(t^2+l^2)^{5/2}}\right],\\
&m_3 = \frac{\Phi_B'''(0)}{1\cdot 2\cdot 3} = -\frac{1}{t^2}\left[m_2 + \frac{5}{16\,(t^2+l^2)^{7/2}}\right],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&m_n = \frac{\Phi_B^{(n)}(0)}{n!} = -\frac{1}{t^2}\left[m_{(n-1)} + (-1)^{(n+1)}\frac{(2n-1)!!}{2^n n!\,(t^2+l^2)^{(2n+1)/2}}\right].
\end{aligned}\right\}\ \ldots\ (II.5)$$

Using the expressions (II.1) and (II.5), we may write the formulae of X_l and Z_l in the forms:

$$\left.\begin{aligned}
&X_l = -2\frac{l}{\sqrt{t^2+l^2}}\cdot\frac{\varepsilon M_{/\!/}x+\varepsilon M_{\perp}t}{x^2+t^2} + h_0 + h_1x + h_2x^2 + h_3x^3 + \ldots\\
&\text{where}\\
&\quad h_0 = 2\,n_0\,l\,\varepsilon M_{\perp}\sin\varphi,\\
&\quad h_1 = 2\,n_1\,l\,\varepsilon M_{\perp}\sin\varphi,\\
&\quad h_2 = 2\,l\,\varepsilon M_{\perp}(-tm_1 + n_2\sin\varphi),\\
&\quad h_3 = 2\,l\,(-m_1\varepsilon M_{/\!/} + n_3\,\varepsilon M_{\perp}\sin\varphi),\\
&\quad h_4 = 2\,l\,\varepsilon M_{\perp}(-tm_2 + n_4\sin\varphi),\\
&\quad \ldots\ldots\ldots\ldots
\end{aligned}\right\}\ \ldots\ (II.6)$$

and

$$\left.\begin{aligned}
&Z_l = 2\frac{l}{\sqrt{t^2+l^2}}\cdot\frac{\varepsilon M_{/\!/}t-\varepsilon M_{\perp}x}{x^2+t^2} + k_0 + k_1x + k_2x^2 + k_3x^3 + \ldots\\
&\text{where}\\
&\quad k_0 = 2\,n_0\,l\,\varepsilon M_{\perp}\cos\varphi,\\
&\quad k_1 = 2\,n_1\,l\,\varepsilon M_{\perp}\cos\varphi,\\
&\quad k_2 = 2\,l\,(tm_1\varepsilon M_{/\!/} + n_2\,\varepsilon M_{\perp}\cos\varphi),\\
&\quad k_3 = 2\,l\,\varepsilon M_{\perp}(-m_1 + n_3\cos\varphi),\\
&\quad k_4 = 2\,l\,(tm_2\varepsilon M_{/\!/} + n_4\,\varepsilon M_{\perp}\cos\varphi),\\
&\quad \ldots\ldots\ldots\ldots
\end{aligned}\right\}\ \ldots\ (II.7)$$

Put

$$\left.\begin{aligned} X_r &= h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \dots \\ Z_r &= k_0 + k_1 x + k_2 x^2 + k_3 x^3 + \dots \\ M_{//}' &= \frac{l}{\sqrt{t^2 + l^2}} M_{//},\ M_{\perp}' = \frac{l}{\sqrt{t^2 + l^2}} M_{\perp}. \end{aligned}\right\} \dots\dots\dots \quad (II.8)$$

Further, let X_∞ and Z_∞ denote the values of X_l and Z_l in the case where the edge is infinite. Let X'_∞ and Z'_∞ correspond to the same sheet of infinite length, the edge- and cross-magnetizations being diminished to $M_{//}'$ and $M_{\perp}'$.

Using these symbols, formulae (*II.3, II.4*) in case *A* and formulae (*II.6—II.7*) in case *B* can be written

A. $x^2 < l^2 - t^2$.

$$\left.\begin{aligned} X_l &= X_\infty + X_r, \\ Z_l &= Z_\infty + Z_r. \end{aligned}\right\} \dots\dots\dots \quad (II.9)$$

B. $x^2 < t^2$.

$$\left.\begin{aligned} X_l &= X'_\infty + X_r, \\ Z_l &= Z'_\infty + Z_r. \end{aligned}\right\} \dots\dots\dots \quad (II.10)$$

When l is $> t\sqrt{2}$, the formulae (*II.9*) are valid in a greater interval of x than the formulae (*II.10*). When l is $< t\sqrt{2}$, the formulae (*II.10*) have the greater interval of validity.

It may be pointed out that the potence series X_r and Z_r are not identical in the cases A and B, as the coefficients of these potence series are different in the two mentioned cases (comp. equations *II.3—II.4* and *II.6—II.7*).

§ 12. Generalisations. Interpretation equation.

The formulae given in the preceding section are valid for the plane perpendicular to and across the middle of the edge. Further they presume that $x_0 = 0$ and that the angles α, β, γ are zero.

If in the expressions of X_∞, X'_∞, X_r, Z_∞, Z'_∞, Z_r we exchange x in $(x - x_0)$, the formulae are also valid when x_0 is different from zero. In this case X_∞, X'_∞ and Z_∞, Z'_∞ assume the forms given in (*5.1*) and (*5.2*) and X_r and Z_r are still potence series of x, but their coefficients are changed. The interval of validity of these potence series is obviously in case A determined by the relation $(x - x_0)^2 < l^2 - t^2$ and in the case B by the relation $(x - x_0)^2 < t^2$.

In a plane, parallel to the plane of symmetry mentioned above and cutting the edge at the distances l_1 and l_2 from its ends, the X- and Z-components $(X_{l_1 l_2}, Z_{l_1 l_2})$ are influenced by the length magnetization $(M_{=})$ of the sheet. If $M_{=}$ is zero, or if the plane is situated so close to the middle of the edge that the influence of the length magnetization can be omitted, we have

A. $X_{l_1 l_2} = \frac{1}{2}(X_{l_1} + X_{l_2}) = X_\infty + \frac{1}{2}(X_{r_1} + X_{r_2})$,

B. $X_{l_1 l_2} = \frac{1}{2}(X_{l_1} + X_{l_2}) = X'_\infty + \frac{1}{2}(X_{r_1} + X_{r_2})$,

where in case B $\quad \dfrac{M_{//}'}{M_{//}} = \dfrac{M_{\perp}'}{M_{\perp}} = \dfrac{1}{2}\left[\dfrac{l_1}{\sqrt{t^2 + l_1^2}} + \dfrac{l_2}{\sqrt{t^2 + l_2^2}}\right].$

Analogous formulae are obviously valid for $Z_{l_1 l_2}$.

As regards a component F, obviously

$$F_{l_1 l_2} = v_x X_{l_1 l_2} + v_y Y_{l_1 l_2} + v_z Z_{l_2 l_1}.$$

Thus, it is possible to write

A. $F_{l_1 l_2} = F_\infty + F_{r_1 r_2}$

B. $F_{l_1 l_2} = F'_\infty + F_{r_1 r_2}$,

where in the case of $Y_{l_1 l_2} = 0$,

$$F_\infty = v_x X_\infty + v_z Z_\infty,$$

$$F'_\infty = v_x X'_\infty + v_z Z'_\infty,$$

$$F_{r_1 r_2} = v_x \frac{X_{r_1} + X_{r_2}}{2} + v_z \frac{Z_{r_1} + Z_{r_2}}{2}.$$

When $Y_{l_1 l_2}$ is not zero, it is obviously still possible to use the above formulae for $F_{l_1 l_2}$, if $Y_{l_1 l_2}$ can be expressed in a potence series within the interval of validity of X_{r_1} and Z_{r_1} (l_1 presumed $< l_2$) and F_r is substituted for $F_{r_1 r_2}$. Thereby F_r denotes the sum of the two potence series $F_{r_1 r_2}$ and $Y_{l_1 l_2}$.

In the following, we write the potence series

$$F_r = \Delta F + k_F x + k_F' x^2 + k_F'' x^3 + k_F''' x^4 \quad \ldots\ldots\ldots\ldots\ldots \quad (12.1)$$

From the preceding discussion one can state, that in the case $\alpha = \beta = \gamma = 0$, the coefficients ΔF, k_F', k_F'' ... of F_r are constant along a profile line parallel to the x-axis, and situated within or in the vicinity of the plane of symmetry. To different profile lines correspond, however, different values of the coefficients of the potence series, since these quantities are functions of t and the position of the profile line in relation to the plane of symmetry. Furthermore a displacement of the zero-point of the profile line changes the values of the mentioned coefficients.

It is possible to establish that, even in the case where α, β and γ are $\neq 0$, the component F along a profile line can be expressed in formulae analogous to those of A and B in the case: $\alpha = \beta = \gamma = 0$. Thus, it can be stated that a component F along a profile line, situated in a vertical plane across the middle (or in the vicinity of the middle) of a sheet of finite length, can be written

$$F = F_\infty + F_r,$$

within a certain interval around the head point. In the mentioned expression F_∞ symbolises the formula, valid for a sheet of infinite length, and F_r a potence series according to (*12.1*). The infinite sheet has the same position as the sheet of finite length, but the magnetization of the former sheet may be smaller. The ratio between the edge magnetization ($M_{//}$) and the cross magnetization ($M_\perp$), however, is probably identical for the two sheets.

The methods of interpretation, given earlier for a sheet of infinite length, can now be extended to a sheet of finite length.

Using the first n terms of the potence series F_r, we can write

$$F_\infty = F - F_r = F - \Delta F - k_F x - k_F' x^2 - k_F'' x^3 \ldots - k_F^{(n-2)} x^{n-1} \quad \ldots\ldots \quad (12.2)$$

Inserting this expression of F_∞ in the interpretation equation (*6.1*), this equation can be written

$$A_0 + A_1 x + A_2 x^2 + \ldots + A_{n+1} x^{n+1} + b_0 F + b_1 x F = x^2 F, \qquad (12.3)$$

where

$$\left.\begin{aligned} A_0 &= a_0 - b_0 \Delta F, \\ A_1 &= a_1 - b_1 \Delta F - b_0 k_F, \\ A_2 &= \Delta F - b_1 k_F - b_0 k_F', \\ A_3 &= k_F - b_1 k_F' - b_0 k_F'', \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_n &= k_F^{(n-3)} - b_1 k_F^{(n-2)} \\ A_{n+1} &= k_F^{(n-2)} \end{aligned}\right\} \qquad (12.4)$$

The formula (*12.3*) can be used as *interpretation equation*, as regards a sheet of finite length and one given component F. This implies, that the way to follow in the interpretation calculations is the same as that followed in the case of an infinite sheet.

The first step is to determine the four parameters a_0, a_1, b_0, b_1, corresponding to the infinite sheet, and the n coefficients ΔF, k_F, k_F' ... $k_F^{(n-2)}$ of the potence series F_r. The data required for this purpose are taken from $(n + 4)$ datum points selected at a reference line in the way shown in Fig. 5.1. The practical procedure to solve the parameters and coefficients mentioned above from the system of interpretation equations, corresponding to these datum points, and the relations (*12.4*) is treated later on in § 16.

The second step in the calculations is to compute the *magnetic parameters* of the sheet. Thereby the values of x_0, t_0, $\varepsilon M_{//}'$ and $\varepsilon M_\perp'$ are obtained direct from the formulae (7.7), by inserting the computed values of a_0, a_1, b_0 and b_1. For the edge magnetization ($M_{//}$) and the cross magnetization ($M_\perp$) of the finite sheet the following is valid:

$$1 \leqslant \varepsilon M_{//} : \varepsilon M_{//}' = \varepsilon M_\perp : \varepsilon M_\perp' \leqslant \frac{\sqrt{t_0^2 + l_1^2}}{l_1},$$

where l_1 is the distance between the vertical plane through the reference line and the nearest end of the edge.

From a theoretical point of view it is possible to compute the dip angle (φ) and the length of the edge by using the values of ΔF, k_F, k_F' ... It is not probable, however, that such calculations will give any real results except in isolated cases; this being due to facts, that are going to be dealt with in the next section.

The calculation of the magnetic parameters of a sheet, where two or three components are given, is being treated in § 17 and § 18.

§ 13. Strange disturbing fields.

The magnetic profile curves given for the interpretation are very often influenced by fields from other magnetic bodies and masses than the actual sheet. We divide these strange disturbing anomalies into two groups: 1) regional anomalies, and anomalies caused by tectonic structures or bodies of about the same, or essentially larger dimensions than the actual sheet, and 2) anomalies generated by small local magnetic masses in the immediate vicinity of a point of measurement. We call an anomaly according to 2) "punctual anomaly". It affects in general only a small area around the actual point and then varies very rapidly in all directions from that point.

The magnetic profile curves are distorted by observation errors. The sum of the punctual anomaly and the observation error in a point at a profile curve (F) is in the following called "punctual error".

From the facts given above it appears that we may write

$$F = F_s + F_d + F_p,$$

where F_s is caused by the sheet, F_d is the disturbing field of large dimensions and F_p is the "punctual error".

If F_d, within the interval occupied by the datum points, may be represented with sufficient accuracy by a potence series and if F_p can be omitted, then, obviously, the interpretation equation (*12.3*) is valid. In this case the remainder field F_r is composed of the difference field ($F_s - F_\infty$) and the strange disturbing field F_d.

In the case where a sheet-like body can be considered to be of infinite length, *i. e.* ($F_s - F_\infty$) can be omitted, it is obviously possible to escape the influence of the disturbing field F_d in the calculated results, by using the interpretation equation (*12.3*) instead of (*6.1*).

In practical interpretation work we always have to reckon with the possibility that a sheet (A) cannot be considered as having an infinite extent in depth. As mentioned earlier (§ 2) the field from a sheet (A) of finite extent in depth is equivalent with the fields from two sheets (B) and (C) of infinite extent in depth. Thereby F_s may correspond to the sheet (B), and the field from sheet (C), which has its edge at the lower edge of (A), may be contained in F_d. This part of F_d can obviolusly be represented with sufficient accuracy by a potence series regarding x, valid within the actual interpretation interval, when the distance (d) between this stretch on the reference line and the edge of the sheet (C) exceeds a certain minimum value (d_m). Consequently, it is possible to use the interpretation equation (*12.3*) if d is $> d_m$. When d is $< d_m$, it is necessary to use the interpretation equation (*20.1*) of two sheets given in chapter V.

The author has found that it is often more advantageous to use the interpretation equation of one sheet than that of two sheets by calculating the depth of a body, when the distance d is essentially greater than d_m.

If F_p is an essential part of F, the errors in the calculated values of $A_0, A_1 \ldots$, b_0, b_1 (and consequently in the magnetic parameters of the sheet) may be large when using the interpretation equation (*12.3*) for a single set of datum points. The punctual errors, however, existing at the points along the reference line oscillate about zero in a case of chance. Consequently, it is possible to eliminate to a great extent the mentioned effect of the punctual errors. This is e. g. the case if we compute by means of the method of least squares, the probable values of the parameters of equation (*12.3*), by using a great amount of data from the *F*-curve within a suitable selected interval of the profile line. However, the introduction of integral functions of the *F*-curve in the calculations will prove much better than the employment of the time-vasting method of least squares.

§ 14. Interpretation equations containing integral functions of the magnetic profile curves.

Multiply the equation (*12.3*) with x^p and integrate this expression with regard to x over the interval $x_b < x < x_a$; then, excluding the value $p = -1$ the following formula is obtained

$$\left.\begin{aligned} & A_0 \frac{x_a^{p+1} - x_b^{p+1}}{p+1} + A_1 \frac{x_a^{p+2} - x_b^{p+2}}{p+2} + \ldots + A_{n+1} \frac{x_a^{p+n+2} - x_b^{p+n+2}}{p+n+2} + \\ & \qquad + b_0 F(p)_{x_b}^{x_a} + b_1 F(p+1)_{x_b}^{x_a} = F(p+2)_{x_b}^{x_a}, \\ & \text{where } F(p)_{x_b}^{x_a} = \int_{x_b}^{x_a} x^p F\,dx, \quad F(p+1)_{x_b}^{x_a} = \int_{x_b}^{x_a} x^{p+1} F\,dx, \quad \text{a. s. o.} \end{aligned}\right\} \ldots (14.1)$$

The expression above can be considered as the general integral form of the interpretation equation (*12.3*). When using this formula it is obviously necessary to know F all the way between x_b and x_a along the reference line and not only at some datum points as in the case shown in Fig. *5.1*. The quantities $F(p)_{x_b}^{x_a}$, $F(p+1)_{x_b}^{x_a}$, $F(p+2)_{x_b}^{x_a}$ can be determined by drawing up the curves of x^pF, $x^{p+1}F$, $x^{p+2}F$ and integrating these curves over the interval $x_b < x < x_a$ with a planimeter.

When calculating the parameters of equation (*14.1*) we have in principle the choice of three different ways, viz. to apply the mentioned equation for a) a number of p-values or b) a number of intervals or c) to use a combination of the methods a) and b). In the following, formulae are given for some special cases belonging to b) and c).

1) Intervals situated symmetrical in relation to the zero point at the reference line.

For two symmetrical intervals, one situated between the points x_i and x_{i+1} and the other between the points $-x_i$ and $-x_{i+1}$ at the reference line, the two formulae below can be derived from (*14.1*).

$$\left.\begin{aligned}
&A_0(x_{i+1}^{p+1}-x_i^{p+1})+\frac{p+1}{p+3}A_2(x_{i+1}^{p+3}-x_i^{p+3})+\frac{p+1}{p+5}A_4(x_{i+1}^{p+5}-x_i^{p+5})+\ldots \\
&\quad+\frac{p+1}{2}b_0\left[F(p)_{x_i}^{x_{i+1}}+(-1)^p F(p)_{-x_{i+1}}^{-x_i}\right]+\frac{p+1}{2}b_1\left[F(p+1)_{x_i}^{x_{i+1}}+\right. \\
&\left.+(-1)^p F(p+1)_{-x_{i+1}}^{-x_i}\right]=\frac{p+1}{2}\left[F(p+2)_{x_i}^{x_{i+1}}+(-1)^p F(p+2)_{-x_{i+1}}^{-x_i}\right], \\
&A_1(x_{i+1}^{p+2}-x_i^{p+2})+\frac{p+2}{p+4}A_3(x_{i+1}^{p+4}-x_i^{p+4})+\frac{p+2}{p+6}A_5(x_{i+1}^{p+6}-x_i^{p+6})+\ldots \\
&+\frac{p+2}{2}b_0\left[F(p)_{x_i}^{x_{i+1}}-(-1)^p F(p)_{-x_{i+1}}^{-x_i}\right]+\frac{p+2}{2}b_1\left[F(p+1)_{x_i}^{x_{i+1}}-\right. \\
&\left.-(-1)^p F(p+1)_{-x_{i+1}}^{-x_i}\right]=\frac{p+2}{2}\left[F(p+2)_{x_i}^{x_{i+1}}-(-1)^p F(p+2)_{-x_{i+1}}^{-x_i}\right].
\end{aligned}\right\}\ldots (14.2)$$

2) The zero point of the reference line is chosen in the middle of the interval $x_b < x < x_a$, *i. e.* $x_b = -x_a$.

Putting $x_{i+1} = x_a$ and $x_i = 0$ in the formulae (*14.2*), it is possible to write these equations

$$\left.\begin{aligned}
&A_0+\frac{p+1}{p+3}A_2x_a^2+\frac{p+1}{p+5}A_4x_a^4+\ldots+\frac{p+1}{2x_a^{p+1}}b_0\left[F(p)_0^{x_a}+(-1)^pF(p)_{-x_a}^0\right]+ \\
&\quad+\frac{p+1}{2x_a^{p+1}}b_1\left[F(p+1)_0^{x_a}+(-1)^p F(p+1)_{-x_a}^0\right]= \\
&\quad=\frac{p+1}{2x_a^{p+1}}\left[F(p+2)_0^{x_a}+(-1)^p F(p+2)_{-x_a}^0\right], \\
&A_1+\frac{p+2}{p+4}A_3x_a^2+\frac{p+2}{p+6}A_5x_a^4+\ldots+\frac{p+2}{2x_a^{p+2}}b_0\left[F(p)_0^{x_a}-(-1)^pF(p)_{-x_a}^0\right]+ \\
&\quad+\frac{p+2}{2x_a^{p+2}}b_1\left[F(p+1)_0^{x_a}-(-1)^p F(p+1)_{-x_a}^0\right]= \\
&\quad=\frac{p+2}{2x_a^{p+2}}\left[F(p+2)_0^{x_a}-(-1)^p F(p+2)_{-x_a}^0\right].
\end{aligned}\right\}\ldots (14.3)$$

As regards the parameters of the formulae above, it is of interest to observe that without b_0 and b_1 only those with even indices (A_0, $A_2\ldots$) are present in the first equation both in (*14.2*) and (*14.3*) and only those with odd indices (A_1, $A_3\ldots$) in the second equation. This fact obviously simplifies the calculations.

CHAPTER IV.

Technics of Calculation.

§ 15. Introduction.

In the following, a systematical treatment is given of the technics of calculation by using the interpretation methods derived in chapters II and III. Thereby we distinguish the cases where one, two or three components of the anomaly field are known.

In the simplest case, an interpretation calculation includes only the determination of the four magnetic parameters x_0, t_0, $\varepsilon M_{/\!/}$ and $\varepsilon M_{\perp}$. In other cases it is desirable also to compute the angles α and γ. Further it is often necessary to pay regard to the remainder fields. Thereby, in the following, the general case is treated where the remainder fields are determined by n coefficients according to (*12.2*).

As regards the case of one given component, the methods given in the preceding are based on the equations (*6.1*) and (*12.3*). The former of these interpretation equations can be considered as a special case of the latter and can be used with success only when the remainder field (F_r) can be omitted. The calculation problem in this case is to devise as rational and rapid methods as possible for computing the parameter functions a_0, a_1, b_0, b_1 and the coefficients ΔF, k_F, k_F' . . . of the remainder field.

When two or three components are given the possibility exists to compute the angles α and γ. Further the problem remains how to compute the remainder fields.

In the following treatment determinants are used to a great extent. Thereby, inter alia, the significations occur:

$$c_F^{(m)} = \left| 1 \quad x_i \quad x_i^2 \quad \ldots\ldots \quad x_i^m \quad F_i \right|,$$

$$c_{x^pF}^{(m)} = \left| 1 \quad x_i \quad x_i^2 \quad \ldots\ldots \quad x_i^m \quad x_i^p F_i \right|,$$

$$c^{(m)} = \left| 1 \quad x_i \quad x_i^2 \quad \ldots\ldots \quad x_i^m \quad x_i^{m+1} \right|,$$

where the determinants are rectangular with $(m + 2)$ rows, corresponding to $(m + 2)$ different indices i.

Further we signify the subdeterminant of the first order of $c^{(m)}$, corresponding to the element in row (j) and column $(m + 2)$ with $c_{j\,(m+2)}^{(m)}$. If the indices of the first row are $i = 1$, of the second $i = 2$ and so on, we may write

$$c_F^{(m)} = \sum_{i=1}^{i=m+2} F_i c_{i\,(m+2)}^{(m)}, \quad c_{x^pF}^{(m)} = \sum_{i=1}^{i=m+2} x_i^p F_i c_{i\,(m+2)}^{(m)},$$

where for $i = j$

$$c^{(m)}_{j\,(m+2)} = (-1)^{j+m}\,(x_2 - x_1)\,(x_3 - x_1)\,\ldots\,(x_{j-1} - x_1)\,(x_{j+1} - x_1)\ldots$$
$$(x_{m+2} - x_1)\,(x_3 - x_2)\,(x_4 - x_2)\,\ldots\,(x_{j-1} - x_2)\,(x_{j+1} - x_2)\,\ldots\,(x_{m+2} - x_2)$$
$$(x_4 - x_3)\,\ldots\ldots\,(x_{m+2} - x_{m+1}) = (-1)^{j+m}\prod (x_i - x_q). \qquad (15.1)$$

constituting the product of all differences $(x_i - x_q)$ for all $i > q$ from 1 to $m + 2$, with $i = j$ left out.

When we use the data of x_i and F_i in $(m + 3)$ points, we let the determinants $c^{(m)}_F$, $c^{(m)}_{xPF}$ and $c^{(m)}$ include the rows corresponding to all $(m + 3)$ i-values except one for $i = h$ and signify by $d^{(m)}_F$, $d^{(m)}_{xPF}$ and $d^{(m)}$ the determinants mentioned when they include the rows corresponding to all $(m + 3)$ i-values except one for $i = k$. Further we then let $c^{(m)}_{j(m+2)}$ and $d^{(m)}_{j(m+2)}$ signify the subdeterminants of the first order corresponding to the element in the $(m + 2)$:th column and in the row with index $i = j$.

If F_i is changed in G_i or K_i or some other quantities e_i, f_i, g_i, k_i in the above determinants we signify these determinants by $c^{(m)}_G$, $c^{(m)}_{xPG}$, $c^{(m)}_K$, $c^{(m)}_{xPK}\ldots$ and so on.

§ 16. Given one component (F).

Firstly, some particular facts have to be pointed out. It is important to state that the values of the parameters of the interpretation equation are dependent on the position of the zero point at the reference line. On changing the point mentioned from one point (O) to another point (O'), having $x = d$, the following relations are valid between those parameters corresponding to the new zero point (indicated by a dash) and those corresponding to the original zero point.

Putting $\varphi(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_{n+1} x^{n+1}$,
$\psi(x) = b_0 + b_1 x - x^2$,

we can write

$$\left.\begin{aligned}
A_0' &= \varphi(d) = A_0 + A_1 d + A_2 d^2 + \ldots + A_{n+1} d^{n+1},\\
A_1' &= \frac{\varphi'(d)}{1} = A_1 + 2A_2 d + 3A_3 d^2 + \ldots + (n+1)\,A_{n+1} d^n,\\
A_2' &= \frac{\varphi''(d)}{1\cdot 2} = A_2 + 3A_3 d + 6A_4 d^2 + \ldots\\
A_{n+1}' &= \frac{\varphi^{n+1}(d)}{(n+1)!} = A_{n+1}\\
b_0' &= \psi(d) = b_0 + b_1 d - d^2, \quad b'_1 = \frac{\psi'(d)}{1} = b_1 - 2d.
\end{aligned}\right\} \qquad (16.1)$$

It is often possible to simplify the calculations of the parameters considerably by a suitable choice of zero point at the x-axis and of calculation points on the given F-curve. For example, if we take an extreme point $(x_e F_e)$ as calculation point, we find by differentiating the interpretation equation that

$$A_1 + 2A_2x_e + 3A_3x_e^2 + \ldots + (n+1)\,A_{n+1}x_e^n + b_1F_e = 2\,x_eF_e.$$

Putting $x_e = 0$, the extreme point gives us the two relations

$$A_0 + b_0F_e = 0.$$
$$A_1 + b_1F_e = 0.$$

In treating actual interpretation problems, the possibility of choosing the positions of the calculation points from a point of view of facilitating the calculations varies widely from case to case. The field measurements available may have been made at such distances from each other in the terrain that the positions of the extreme points on the F-curve cannot be fixed with sufficient accuracy. Also magnetic fields from disturbing bodies in the neighbourhood of the sheet may so distort the magnetic picture that only F-values from limited portions of a measured profile line can form the basis of the interpretation calculations. The qualification that the basic points are to refer to points in the terrain lying along a straight line, may restrict the choice of calculation points, if the ground is broken. It thus appears from the foregoing that the technique of calculating the parameters must be suited to the nature of the observation data at hand.

Simple cases.

Below there are treated some cases of great importance for the practical work.

1) *Four calculation points* $(x_1F_1,\ x_1'F_1,\ x_2F_2,\ x_2'F_2)$ *having pairwise the same F-value.*

Since the interpretation equation (6.1) is of the second degree in x and for $F = F_1$, and $F = F_2$ must be satisfied by x_1 and x_1' respectively x_2 and x_2', we may write

$$F_1(x - x_1)\,(x - x_1') = x^2F_1 - x(a_1 + b_1F_1) - b_0F_1 - a_0,$$
$$F_2(x - x_2)\,(x - x_2') = x^2F_2 - x(a_1 + b_1F_2) - b_0F_2 - a_0.$$

From this we obtain by equating the coefficients for x in the left and right membra

$$\left.\begin{aligned} a_0 + b_0F_1 &= -\,x_1x_1'F_1, & a_1 + b_1F_1 &= (x_1 + x_1')F_1, \\ a_0 + b_0F_2 &= -\,x_2x_2'F_2, & a_1 + b_1F_2 &= (x_2 + x_2')F_2. \end{aligned}\right\} \ldots \quad (16.2)$$

The parameters may hence be expressed

$$\left.\begin{aligned} a_0 &= -\frac{(x_1x_1' - x_2x_2')F_1F_2}{F_2 - F_1}, & b_0 &= \frac{x_1x_1'F_1 - x_2x_2'F_2}{F_2 - F_1}, \\ a_1 &= \frac{(x_1 + x_1' - x_2 - x_2')F_1F_2}{F_2 - F_1}, & b_1 &= -\frac{(x_1 + x_1')F_1 - (x_2 + x_2')F_2}{F_2 - F_1}. \end{aligned}\right\} \ldots \quad (16.3)$$

Special case: One pair of calculation points coincide in an extreme point, *i. e.* we can put $x_2 = x_2' = x_e$ and $F_2 = F_e$.

Putting $x_e = 0$, we obtain

$$\left.\begin{aligned} a_0 &= -\,b_0F_e = -\frac{x_1x_1'F_1}{F_e - F_1}\,F_e, \\ a_1 &= -\,b_1F_e = \frac{(x_1 + x_1')F_1}{F_e - F_1}\,F_e. \end{aligned}\right\} \quad \text{(16.4)}$$

Obviously, the above formulae are valid for the case where both pairs of the calculation points each coincide with an extreme point, *i. e.* $x_1 = x_1' = x_e'$ and $F_1 = F_e'$.

2) *Two of the calculation points* $(x_1F_1,\ x_1'F_1,\ x_2F_2,\ x_3F_3)$ *have the same F-value.*

As in 1) it is evident that

$$a_0 + b_0F_1 = -\,x_1x_1'F_1, \qquad a_1 + b_1F_1 = (x_1 + x_1')F_1.$$

Putting $u_i = \dfrac{F_i}{F_1 - F_i}\,(x_i - x_1)(x_i - x_1'),$ .... (16.5)

we may write the interpretation equation

$$a_0 + a_1x_i = u_iF_1.$$

Making use of this equation for the points (x_2F_2) and (x_3F_3), one obtains two equations, which give a_0 and a_1. We obtain

$$\left.\begin{aligned} a_0 &= \frac{x_3u_2 - x_2u_3}{x_3 - x_2}\,F_1, \qquad b_0 = -\,x_1x_1' - \frac{a_0}{F_1}, \\ a_1 &= \frac{u_3 - u_2}{x_3 - x_2}\,F_1, \qquad b_1 = x_1 + x_1' - \frac{a_1}{F_1}. \end{aligned}\right\} \quad \text{(16.6)}$$

Special case: The two points having the same F-value coincide in an extreme point (x_eF_e).

We put $x_1 = x_1' = x_e = 0$ and $F_1 = F_e$ in the above formulae and obtain

$$\left.\begin{aligned} a_0 &= -\,b_0F_e = \frac{x_2x_3}{x_3 - x_2}\left(\frac{x_2F_2}{F_e - F_2} - \frac{x_3F_3}{F_e - F_3}\right)F_e, \\ a_1 &= -\,b_1F_e = -\frac{1}{x_3 - x_2}\left(\frac{x_2^2F_2}{F_e - F_2} - \frac{x_3^2F_3}{F_e - F_3}\right)F_e. \end{aligned}\right\} \quad \text{(16.7)}$$

3) *Four datum points* (x_iF_i), *having* $x_4 - x_3 = x_3 - x_2 = x_2 - x_1$.

A straight line $F = F_1$ in the interpretation diagram cuts the F-curve in two points which have the x-coordinates x_1 and x_1'.

We now put

$$n_i = \frac{F_i}{F_1 - F_i}$$

and make use of equation below (16.5), which may be written

$$a_0 + a_1x_i + x_1'(x_i - x_1)n_iF_1 = x_i(x_i - x_1)n_iF_1.$$

From the three equations corresponding to $i = 2, 3, 4$ we obtain

$$x_1' = \frac{n_2 x_2 - 4 n_3 x_3 + 3 n_4 x_4}{n_2^2 - 4 n_3 + 3 n_4}$$

and further

$$\left.\begin{aligned} a_0 &= [x_3(x_2 - x_1')n_2 - 2\, x_2(x_3 - x_1')n_3]F_1, \\ a_1 &= -\, [(x_2 - x_1')n_2 - 2\,(x_3 - x_1')n_3]F_1, \\ b_0 &= -\, x_1 x_1' - \frac{a_0}{F_1}, \\ b_1 &= x_1 + x_1' - \frac{a_1}{F_1}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (16.8)$$

General case.

A profile curve of (F) corresponding to a straight profile line is determined by $(n + 4)$ parameters. These can be given in the form of the n parameters of the remainder field F_r and the four parameters a_0, a_1, b_0 and b_1, which are functions of the six magnetic parameters of the sheet. Thus it is not possible to compute more than four of the six quantities x_0, t_0, $\varepsilon M_{//}$, $\varepsilon M_{\perp}$, α and β with the aid of data from one profile curve of F. For this reason we assume in the following that the angles α and γ are known.

Obviously it is always possible to solve all the parameters of equation (12.3) in the common way out of the system of $(n + 4)$ equations corresponding to $(n + 4)$ chosen data of x_i, F_i, by computing $(n + 5)$ rectangular determinants of $(n + 4)$ rows. Without b_0 and b_1 we then obtain a_0, a_1 and the parameters of F_r with the aid of the expressions (12.4).

It is often better, however, to follow another line of calculation. A system of $(n + 3)$ equations according to (12.3) satisfies the relation

$$\left| 1 \quad x_i \quad x_i^2 \ldots x_i^{n+1} \quad F_i[b_0 + b_1 x_i - x_i^2] \right| = 0$$

As $(n + 4)$ calculation points are given it is possible to select two different relations of the type above. These two relations can be written (pp. 53—54)

$$\left.\begin{aligned} b_0 c_F^{(n+1)} + b_1 c_{xF}^{(n+1)} &= c_{x^2F}^{(n+1)}, \\ b_0 d_F^{(n+1)} + b_1 d_{xF}^{(n+1)} &= d_{x^2F}^{(n+1)}. \end{aligned}\right\}$$

Hence

$$\left.\begin{aligned} b_0 &= \frac{d_{xF}^{(n+1)} c_{x^2F}^{(n+1)} - c_{xF}^{(n+1)} d_{x^2F}^{(n+1)}}{c_F^{(n+1)} d_{xF}^{(n+1)} - d_F^{(n+1)} c_{xF}^{(n+1)}}, \\ b_1 &= \frac{c_F^{(n+1)} d_{x^2F}^{(n+1)} - d_F^{(n+1)} c_{x^2F}^{(n+1)}}{c_F^{(n+1)} d_{xF}^{(n+1)} - d_F^{(n+1)} c_{xF}^{(n+1)}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16.9)$$

After the computation of b_0 and b_1 it is convenient to calculate

$$A_{n+1} = k_F^{(n-2)} = \frac{\left| 1 \;\; x_i \;\; x_i^2 \; .. \; x_i^n \;\; F_i(-b_0 - b_1 x_i + x_i^2) \right|}{\left| 1 \;\; x_i \;\; x_i^2 \; .. \; x_i^n \;\; x_i^{n+1} \right|} =$$

$$= \frac{-b_0 c_F^{(n)} - b_1 c_{xF}^{(n)} + c_{x^2F}^{(n)}}{c^{(n)}} \quad \text{(16.10)}$$

The quantities $k_F^{(n-3)}, \ldots k_F, \Delta F$ may be calculated with the aid of similar formulae.

Putting

$$F_i' = F_i - x_i^{n-1} k_F^{(n-2)},$$
$$F_i'' = F_i' - x_i^{n-2} k_F^{(n-3)},$$
$$\cdots\cdots$$
$$F_i^{(n-1)} = F_i^{(n-2)} - x_i k_F,$$
$$F_i^{(n)} = F_i^{(n-1)} - \Delta F,$$

we obviously have

$$\left.\begin{aligned} k_F^{(n-3)} &= \frac{-b_0 c_{F'}^{(n-1)} - b_1 c_{xF'}^{(n-1)} + c_{x^2F'}^{(n-1)}}{c^{(n-1)}}, \\ k_F^{(n-4)} &= \frac{-b_0 c_{F''}^{(n-2)} - b_1 c_{xF''}^{(n-2)} + c_{x^2F''}^{(n-2)}}{c^{(n-2)}}, \\ &\cdots\cdots \\ \Delta F &= \frac{-b_0 c_{F(n-1)}^{(1)} - b_1 c_{xF(n-1)}^{(1)} + c_{x^2F(n-1)}^{(1)}}{c^{(1)}}, \\ a_0 + a_1 x_i &= F_i^{(n)} (x_i^2 - b_1 x_i - b_0). \end{aligned}\right\} \quad \text{(16.11)}$$

With the aid of the formulae above we may successively calculate $k_F^{(n-3)}$, ... ΔF. Thereby we only need $(n+1)$ values x_i, F_i' by calculation of $k_F^{(n-3)}$, n values of x_i, F_i'' by calculation of $k_F^{(n-4)}$ and so on all the way to 3 values of x_i, $F_i^{(n-1)}$, when calculating ΔF. Finally a_0 and a_1 may be calculated from two equations of the last mentioned form.

§ 17. Given two components (F, G).

We distinguish the two cases when α and γ are both known or both unknown.

α AND γ KNOWN.

Before the treatment of the general case, formulae are derived for a simple case which is often very useful in practical interpretation work.

Simple case.

We presuppose: X and Z given at three datum points (1, 2, 3).

$$X_r = \Delta X, \qquad Z_r = \Delta Z,$$
$$\alpha = \beta = \gamma = 0.$$

Significations:

$$\left.\begin{aligned}
h_i &= \frac{(X_1 - X_i)(x_1 X_1 - x_i X_i) + (Z_1 - Z_i)(x_1 Z_1 - x_i Z_i)}{(X_1 - X_i)^2 + (Z_1 - Z_i)^2},\\
l_i &= -\frac{(x_1 - x_i)(X_1 Z_i - X_i Z_1)}{(X_1 - X_i)^2 + (Z_1 - Z_i)^2},\\
m_i &= \frac{(x_1 - x_i)(X_1 - X_i)}{(X_1 - X_i)^2 + (Z_1 - Z_i)^2}, \quad n_i = \frac{(x_1 - x_i)(Z_1 - Z_i)}{(X_1 - X_i)^2 + (Z_1 - Z_i)^2}.
\end{aligned}\right\} \quad \ldots\ldots (17.1)$$

If $\Delta X = \Delta Z = 0$, we have according to the formulae (6.9) that $x_0 = h_i$ and $t_0 = l_i$.

If the real anomaly values are $(X_i - \Delta X)$ and $(Z_i - \Delta Z)$ we can write

$$\left.\begin{aligned}
x_0 &= h_i + \Delta h_i = h_i - m_i \Delta X - n_i \Delta Z,\\
t_0 &= l_i + \Delta l_i = l_i - n_i \Delta X + m_i \Delta Z.
\end{aligned}\right|$$

For $i = 2$ and 3 we obtain four equations altogether from which ΔX, ΔZ, x_0 and t_0 may be solved. $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ may then be calculated according to equation (6.8). We find that

$$\left.\begin{aligned}
\Delta X &= \frac{(h_2 - h_3)(m_2 - m_3) + (l_2 - l_3)(n_2 - n_3)}{(m_2 - m_3)^2 + (n_2 - n_3)^2},\\
\Delta Z &= \frac{(h_2 - h_3)(n_2 - n_3) - (l_2 - l_3)(m_2 - m_3)}{(m_2 - m_3)^2 + (n_2 - n_3)^2},\\
x_0 &= h_i - m_i \Delta X - n_i \Delta Z,\\
t_0 &= l_i - n_i \Delta X + m_i \Delta Z,\\
2\varepsilon M_{//} &= -(x_i - x_0)(X_i - \Delta X) + t_0(Z_i - \Delta Z),\\
2\varepsilon M_{\perp} &= -t_0(X_i - \Delta X) - (x_i - x_0)(Z_i - \Delta Z).
\end{aligned}\right\} \quad \ldots\ldots\ldots (17.2)$$

General case.

If the remainder fields F_r and G_r each contains (n) parameters we have to calculate $(2n + 4)$ unknown quantities.

Consequently we must use data of F_i and G_i for $(n + 2)$ points x_i on a reference line. As each of the two equations (7.10) contains $(n + 3)$ unknown parameters we may put the determinants

$$\left| 1 \;\; x_i \;\; x_i^2 \ldots x_i^n \;\; [a(A_v G_i - A_u F_i) - b(B_v G_i - B_u F_i) + x_i(B_v G_i - B_u F_i)] \right| = 0,$$
$$\left| 1 \;\; x_i \;\; x_i^2 \ldots x_i^n \;\; [a(B_v G_i - B_u F_i) + b(A_v G_i - A_u F_i) - x_i(A_v G_i - A_u F_i)] \right| = 0.$$

These expressions can be written

$$a\left[A_v c_G^{(n)} - A_u c_F^{(n)}\right] - b\left[B_v c_G^{(n)} - B_u c_F^{(n)}\right] = -B_v c_{xG}^{(n)} + B_u c_{xF}^{(n)},$$
$$a\left[B_v c_G^{(n)} - B_u c_F^{(n)}\right] + b\left[A_v c_G^{(n)} - A_u c_F^{(n)}\right] = A_v c_{xG}^{(n)} - A_u c_{xF}^{(n)}.$$

Solving a and b from these equations and inserting the expressions (7.10) for A_v, A_u, B_v, B_u, we obtain

$$\left.\begin{aligned} a = \frac{t_0}{\mu^2} &= \frac{\left[c_G^{(n)} c_{xF}^{(n)} - c_F^{(n)} c_{xG}^{(n)}\right](v_x' u_z' - v_z' u_x')}{(c_F^{(n)})^2 (u_x'^2 + u_z'^2) + (c_G^{(n)})^2 (v_x'^2 + v_z'^2) - 2\, c_F^{(n)} c_G^{(n)} (v_x' u_x' + v_z' u_z')}, \\ b = x_0 - a\,\frac{p \cos\gamma}{\cos\alpha} \\ &= \frac{c_F^{(n)} c_{xF}^{(n)} (u_x'^2 + u_z'^2) + c_G^{(n)} c_{xG}^{(n)} (v_x'^2 + v_z'^2) - \left[c_G^{(n)} c_{xF}^{(n)} + c_F^{(n)} c_{xG}^{(n)}\right](v_x' u_x' + v_z' u_z')}{(c_F^{(n)})^2 (u_x'^2 + u_z'^2) + (c_G^{(n)})^2 (v_x'^2 + v_z'^2) - 2\, c_F^{(n)} c_G^{(n)} (v_x' u_x' + v_z' u_z')}, \end{aligned}\right\} \quad (17.3)$$

where the quantities v_x', v_z', u_x', u_z', p and μ^2 can be computed with the aid of the expressions given in (7.6).

Observing that

$$-b_0 = a^2 + b^2,$$
$$b_1 = 2b,$$

we may use the equation (16.10) by calculation of the quantities ΔF, k_F, k_F' ... $k_F^{(n-2)}$. We obtain

$$k_F^{(n-2)} = \frac{\left|1 \;\; x_i \;\; x_i^2 \ldots x_i^n \;\; [-b_0 F_i - b_1 x_i F_i + x_i^2 F_i]\right|}{\left|1 \;\; x_i \;\; x_i^2 \ldots x_i^n \;\; x_i^{n+1}\right|} = \frac{-b_0 c_F^{(n)} - b_1 c_{xF}^{(n)} + c_{x^2F}^{(n)}}{c^{(n)}}. \quad \ldots\ldots\ldots\ldots (17.4)$$

In a similar way we obtain

$$k_G^{(n-2)} = \frac{-b_0 c_G^{(n)} - b_1 c_{xG}^{(n)} + c_{x^2G}^{(n)}}{c^{(n)}}.$$

Putting $F_i' = F_i - x_i^{n-1} k_F^{(n-2)}$, $\quad G_i' = G_i - x_i^{n-1} k_G^{(n-2)}$,

$$F_i'' = F_i' - x_i^{n-2} k_F^{(n-3)}, \quad G_i'' = G_i' - x_i^{(n-2)} k_G^{(n-3)},$$

and so on all the way to

$$F_i^{(n-1)} = F_i^{(n-2)} - x_i\, k_F, \qquad G_i^{(n-1)} = G_i^{(n-2)} - x_i k_G,$$

we then obviously may write

$$k_F^{(n-3)} = \frac{-b_0 c_{F'}^{(n-1)} - b_1 c_{xF'}^{(n-1)} + c_{x^2F'}^{(n-1)}}{c^{(n-1)}},$$

$$k_G^{(n-3)} = \frac{-b_0 c_{G'}^{(n-1)} - b_1 c_{xG'}^{(n-1)} + c_{x^2G'}^{(n-1)}}{c^{(n-1)}},$$

$$k_F^{(n-4)} = \frac{-b_0 c_{F''}^{(n-2)} - b_1 c_{xF''}^{(n-2)} + c_{x^2F''}^{(n-2)}}{c^{(n-2)}},$$

$$\left.\begin{aligned} k_G^{(n-4)} &= \frac{-b_0 c_{G''}^{(n-2)} - b_1 c_{xG''}^{(n-2)} + c_{x^2G''}^{(n-2)}}{c^{(n-2)}}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \Delta F &= \frac{-b_0 c_{F(n-1)}^{(1)} - b_1 c_{xF(n-1)}^{(1)} + c_{x^2F(n-1)}^{(1)}}{c^{(1)}}, \\ \Delta G &= \frac{-b_0 c_{G(n-1)}^{(1)} - b_1 c_{xG(n-1)}^{(1)} + c_{x^2G(n-1)}^{(1)}}{c^{(1)}}. \end{aligned}\right\} \quad \dots\dots\dots\dots (17.5)$$

With the aid of the formulae above we may successively calculate $k_F^{(n-2)}$, $k_F^{(n-3)}$ ... ΔF and $k_G^{(n-2)}$, $k_G^{(n-3)}$ ... ΔG. Thereby we only use $(n+1)$ values of x_i, F_i', G_i' by calculation of $k_F^{(n-3)}$ and $k_G^{(n-3)}$, n values of x_i, F_i'', G_i'' by calculation of $k_F^{(n-4)}$ and $k_G^{(n-4)}$ and so on all the way to 3 values of x_i, $F_i^{(n-1)}$, $G_i^{(n-1)}$ when calculating ΔF and ΔG.

Finally we calculate $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ by inserting values in the formulae (7.10) corresponding to one i-number of the quantities $F_i^{(n)} = (F_i^{(n-1)} - \Delta F)$, $G_i^{(n)} = (G_i^{(n-1)} - \Delta G)$ and further t_i' and x_i', computed from the expressions (7.9).

α AND γ UNKNOWN.

In this case we have $(2n+6)$ unknown quantities. Consequently we must use the values of F_i and G_i from $(n+3)$ datum points.

We apply the equation (12.3) to F and G and write

$$A_0 + A_1 x_i + A_2 x_i^2 + \dots + A_{n+1} x_i^{n+1} + b_0 F_i + b_1 x_i F_i = x_i^2 F_i,$$
$$A_0' + A_1' x_i + A_2' x_i^2 + \dots + A_{n+1}' x_i^{n+1} + b_0' G_i + b_1' x_i G_i = x_i^2 G_i.$$

Since b_0, b_1, b_0', b_1' are independent of the direction of the field components (see 7.6) and belong to the same datum line, it is obvious that $b_0 = b_0'$ and $b_1 = b_1'$. As each of the two equations above contain $(n+4)$ parameters we obtain by eliminating A_0, A_1 ... A_{n+1} from the system corresponding to the first equation and A_0', A_1' ... A_{n+1}' from the system corresponding to the second equation

$$b_0 c_F^{(n+1)} + b_1 c_{xF}^{(n+1)} = c_{x^2F}^{(n+1)},$$
$$b_0 c_G^{(n+1)} + b_1 c_{xG}^{(n+1)} = c_{x^2G}^{(n+1)}.$$

Hence

$$b_0 = \frac{c_{xG}^{(n+1)} c_{x^2F}^{(n+1)} - c_{xF}^{(n+1)} c_{x^2G}^{(n+1)}}{c_F^{(n+1)} c_{xG}^{(n+1)} - c_G^{(n+1)} c_{xF}^{(n+1)}}, \quad b_1 = \frac{c_F^{(n+1)} c_{x^2G}^{(n+1)} - c_G^{(n+1)} c_{x^2F}^{(n+1)}}{c_F^{(n+1)} c_{xG}^{(n+1)} - c_G^{(n+1)} c_{xF}^{(n+1)}}. \quad \dots\dots\dots (17.6)$$

After the computation of b_0 and b_1 we may obtain $k_F^{(n-2)}$, $k_F^{(n-3)}$, ... ΔF and $k_G^{(n-2)}$, $k_G^{(n-3)}$, ... ΔG in the same way and with the aid of the same formulae as in the case where the angles α and γ are known.

Further we compute, a_0, a_0', a_1 and a_1' with the aid of the equations

$$\left.\begin{aligned} a_0 + a_1 x_i &= F_i^{(n)}(x_i^2 - b_1 x_i - b_0), \\ a_0' + a_1' x_i &= G_i^{(n)}(x_i^2 - b_1 x_i - b_0). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (17.7)$$

The expressions of $\varepsilon M_{/\!/}$ and $\varepsilon M_{\perp}$ according to (7.7) must give the same result if we insert the values of a_0, a_1 or a_0', a_1'. Thus, α and γ are determined by two conditions. These may be written

$$\left.\begin{aligned} &\frac{cv_x' + a_1 v_z'}{v'^2_x + v'^2_z} = \frac{c' u_x' + a_1' u_z'}{u'^2_x + u'^2_z}, \\ &\frac{cv_z' - a_1 v_x'}{v'^2_x + v'^2_z} = \frac{c' u_z' - a_1' u_x'}{u'^2_x + u'^2_z}, \\ &\text{where } c = \frac{2a_0 + a_1 b_1}{\overset{+}{(-)}\sqrt{-4b_0 - b_1^2}}, \quad c' = \frac{2a_0' + a_1' b_1}{\overset{+}{(-)}\sqrt{-4b_0 - b_1^2}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (17.8)$$

In the expressions of c and c' the (—) sign in front of the root marks corresponds to negative values of t_0.

If the first and second equations above are squared and added, we have

$$\frac{v'^2_x + v'^2_z}{u'^2_x + u'^2_z} = \frac{c^2 + a_1^2}{c'^2 + a'^2_1} = m^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17.9)$$

On inserting this expression as the expressions for v_x', v_z', u_x', u_z' according to (7.5) in the equations (17.8) and putting

$$\left.\begin{aligned} A_x &= a_1 v_x - a_1' m^2 u_x, & C_x &= cv_x - c' m^2 u_x, \\ A_y &= a_1 v_y - a_1' m^2 u_y, & C_y &= cv_y - c' m^2 u_y, \\ A_z &= a_1 v_z - a_1' m^2 u_z, & C_z &= cv_z - c' m^2 u_z, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (17.10)$$

we obtain

$$\left.\begin{aligned} A_x \tan\alpha \sin\gamma - A_y \sin\gamma + A_z \frac{\cos\gamma}{\cos\alpha} + C_y \tan\alpha &= -C_x, \\ C_x \tan\alpha \sin\gamma - C_y \sin\gamma + C_z \frac{\cos\gamma}{\cos\alpha} - A_y \tan\alpha &= A_x. \end{aligned}\right\} \quad \ldots\ldots \quad (17.11)$$

From these equations α and γ can be calculated.

If F and G both are parallel to the xz-plane through origo, we have $A_y = C_y = 0$ and

$$\tan\alpha \sin\gamma = -\frac{A_x A_z + C_x C_z}{A_x C_z - A_z C_x}, \quad \frac{\cos\gamma}{\cos\alpha} = \frac{A_x^2 + C_x^2}{A_x C_z - A_z C_x}. \quad \ldots\ldots \quad (17.12)$$

If F and G both are parallel to the yz-plane through origo, we have $A_x = C_x = 0$ and

$$\cos\alpha \tan\gamma = \frac{A_y A_z + C_y C_z}{A_y^2 + C_y^2}, \quad \frac{\sin\alpha}{\cos\gamma} = \frac{A_y C_z - A_z C_y}{A_y^2 + C_y^2}. \quad \ldots\ldots\ldots\ldots \quad (17.13)$$

If F and G both are parallel to the xy-plane through origo, we have $A_z = C_z = 0$ and

$$\left.\begin{aligned} -\sin\gamma &= \tfrac{1}{2}\,\frac{A_x^2 + A_y^2 + C_x^2 + C_y^2}{A_x C_y - A_y C_x}\left[1 \pm \sqrt{1 - 4\left(\frac{A_x C_y - A_y C_x}{A_x^2 + A_y^2 + C_x^2 + C_y^2}\right)^2}\,\right], \\ -\tan\alpha &= \frac{A_x + C_y \sin\gamma}{A_y - C_x \sin\gamma}. \end{aligned}\right\} \quad \ldots (17.14)$$

If the value of $\sin\gamma$ is real we obviously have $\left|\frac{A_x^2 + A_y^2 + C_x^2 + C_y^2}{A_x C_y - A_y C_x}\right| \geqslant 2$. Further only the (—) sign in front of the root mark will give a value between -1 and 1.

§ 18. Given three components (F, G, K).

When three components of the anomaly field are known in two points at a profile line we can compute the six magnetic parameters x_0, t_0, $\varepsilon M_{//}$, $\varepsilon M_{\perp}$, α and γ. If it is necessary to assume the existence of the remainder fields F_r, G_r, K_r, each determined by n parameters, we thus need $(n + 2)$ datum points at the profile line for interpretation.

First we derive suitable formulae for calculation of α and γ. For this reason we apply the equation (7.5) to F, G and K for a point x_i and thus obtain the equations

$$\begin{aligned} 2\varepsilon M_{//}(t_i' v_z' - x_i' v_x') - 2\varepsilon M_{\perp}(t_i' v_x' + x_i' v_z') - F_i(t_i'^2 + x_i'^2) &= 0, \\ 2\varepsilon M_{//}(t_i' u_z' - x_i' u_x') - 2\varepsilon M_{\perp}(t_i' u_x' + x_i' u_z') - G_i(t_i'^2 + x_i'^2) &= 0, \\ 2\varepsilon M_{//}(t_i' s_z' - x_i' s_x') - 2\varepsilon M_{\perp}(t_i' s_x' + x_i' s_z') - K_i(t_i'^2 + x_i'^2) &= 0. \end{aligned}$$

We assume in the following that $\cos\alpha$ and $\cos\gamma$ are both different from zero. Then $(t_i'^2 + x_i'^2) = 0$ only if $t_0 = 0$ and $x_i = x_0$ that is at a point on the edge of the plate, where all field-components are either ∞ or 0. Thus we have $(t_i'^2 + x_i'^2) \neq 0$ for all practical cases of interpretation.

If neither $\varepsilon M_{//}$, $\varepsilon M_{\perp}$ nor $(t_i'^2 + x_i'^2)$ are 0, we must have the determinant

$$\begin{vmatrix} (t_i' v_z' - x_i' v_x') & -(t_i' v_x' + x_i' v_z') & -F_i \\ (t_i' u_z' - x_i' u_x') & -(t_i' u_x' + x_i' u_z') & -G_i \\ (t_i' s_z' - x_i' s_x') & -(t_i' s_x' + x_i' s_z') & -K_i \end{vmatrix} = 0.$$

This equation can be written

$$F_i\,(u_x' s_z' - u_z' s_x') + G_i\,(s_x' v_z' - s_z' v_x') + K_i\,(v_x' u_z' - v_z' u_x') = 0 \quad \ldots\ldots (18.1)$$

According to (7.6) we have

$$\begin{aligned} u_x' s_z' - u_z' s_x' &= -\sin\gamma\,(u_x s_y - u_y s_x) + \cos\alpha\cos\gamma\,(u_x s_z - u_z s_x) \\ &\quad + \sin\alpha\cos\gamma\,(u_y s_z - u_z s_y), \\ s_x' v_z' - s_z' v_x' &= -\sin\gamma\,(s_x v_y - s_y v_x) + \cos\alpha\cos\gamma\,(s_x v_z - s_z v_x) \\ &\quad + \sin\alpha\cos\gamma\,(s_y v_z - s_z v_y), \\ v_x' u_z' - v_z' u_x' &= -\sin\gamma\,(v_x u_y - v_y u_x) + \cos\alpha\cos\gamma\,(v_x u_z - v_z u_x) \\ &\quad + \sin\alpha\cos\gamma\,(v_y u_z - v_z u_y). \end{aligned}$$

Inserting these expressions in equation (*18.1*) we obtain

$$-\tan\alpha\,[(u_y s_z - u_z s_y)\,F_i + (s_y v_z - s_z v_y)\,G_i + (v_y u_z - v_z u_y)\,K_i]$$
$$+\frac{\tan\gamma}{\cos\alpha}\,[(u_x s_y - u_y s_x)\,F_i + (s_x v_y - s_y v_x)\,G_i + (v_x u_y - v_y u_x)\,K_i]$$
$$= (u_x s_z - u_z s_x)\,F_i + (s_x v_z - s_z v_x)\,G_i + (v_x u_z - v_z u_x)\,K_i. \quad \ldots\ldots\ldots\ldots \quad (18.2)$$

Changing F_i in $(F_i - F_r)$, G_i in $(G_i - G_r)$, K_i in $(K_i - K_r)$ and putting

$$\left.\begin{aligned} f_i &= (u_y s_z - u_z s_y)\,F_i + (s_y v_z - s_z v_y)\,G_i + (v_y u_z - v_z u_y)\,K_i,\\ g_i &= (u_x s_z - u_z s_x)\,F_i + (s_x v_z - s_z v_x)\,G_i + (v_x u_z - v_z u_x)\,K_i,\\ -k_i &= (u_x s_y - u_y s_x)\,F_i + (s_x v_y - s_y v_x)\,G_i + (v_x u_y - v_y u_x)\,K_i,\\ &\text{and}\\ v_F &= -(u_y s_z - u_z s_y)\tan\alpha + (u_x s_y - u_y s_x)\frac{\tan\gamma}{\cos\alpha} - (u_x s_z - u_z s_x),\\ v_G &= -(s_y v_z - s_z v_y)\tan\alpha + (s_x v_y - s_y v_x)\frac{\tan\gamma}{\cos\alpha} - (s_x v_z - s_z v_x),\\ v_K &= -(v_y u_z - v_z u_y)\tan\alpha + (v_x u_y - v_y u_x)\frac{\tan\gamma}{\cos\alpha} - (v_x u_z - v_z u_x). \end{aligned}\right\} \ldots \quad (18.3)$$

we may write the equation (*18.2*)

$$f_i\tan\alpha + k_i\frac{\tan\gamma}{\cos\alpha} + (v_F\,\Delta F + v_G\,\Delta G + v_K\,\Delta K) + x_i\,(v_F k_F + v_G k_G + v_K k_K)$$
$$+ \ldots\ldots + x_i^{n-1}\left(v_F k_F^{(n-2)} + v_G k_G^{(n-2)} + v_K k_K^{(n-2)}\right) = -g_i \quad \ldots\ldots\ldots\ldots \quad (18.4)$$

In the special case where $F = X$; $G = Y$ and $K = Z$, we have $v_x = u_y = s_z = 1$ and $v_y = v_z = u_x = u_z = s_x = s_y = 0$. Then

$$\left.\begin{aligned} f_i &= X_i, & v_F &= -\tan\alpha,\\ g_i &= -Y_i, & v_G &= 1,\\ k_i &= -Z_i, & v_K &= \frac{\tan\gamma}{\cos\alpha}. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (18.5)$$

As we have $(n + 2)$ equations of the form (*18.4*), we can compute the $(n + 2)$ unknown quantities $\tan\alpha$, $\frac{\tan\gamma}{\cos\alpha}$, $(v_F\,\Delta F + v_G\,\Delta G + v_K\,\Delta K)$, $\ldots\,(v_F k_F^{(n-2)} + v_G k_G^{(n-2)} + v_K k_K^{(n-2)})$. Thereby it will often be suitable to make the calculations in the following manner.

Obviously the determinant of $(n + 1)$ rows

$$\left| 1 \;\; x_i \;\; x_i^2 \ldots\ldots x_i^{n-1} \;\; \left[f_i\tan\alpha + k_i\frac{\tan\gamma}{\cos\alpha} + g_i\right]\right| = 0.$$

Thus we may write

$$\left.\begin{aligned} c_{\rm f}^{(n-1)} \tan\alpha + c_{\rm k}^{(n-1)} \frac{\tan\gamma}{\cos\alpha} &= -c_{\rm g}^{(n-1)}, \\ d_{\rm f}^{(n-1)} \tan\alpha + d_{\rm k}^{(n-1)} \frac{\tan\gamma}{\cos\alpha} &= -d_{\rm g}^{(n-1)}, \end{aligned}\right\}$$

Hence

$$\left.\begin{aligned} \tan\alpha &= \frac{c_{\rm k}^{(n-1)} d_{\rm g}^{(n-1)} - c_{\rm g}^{(n-1)} d_{\rm k}^{(n-1)}}{c_{\rm f}^{(n-1)} d_{\rm k}^{(n-1)} - c_{\rm k}^{(n-1)} d_{\rm f}^{(n-1)}}, \\ \frac{\tan\gamma}{\cos\alpha} &= \frac{c_{\rm g}^{(n-1)} d_{\rm f}^{(n-1)} - c_{\rm f}^{(n-1)} d_{\rm g}^{(n-1)}}{c_{\rm f}^{(n-1)} d_{\rm \kappa}^{(n-1)} - c_{\rm k}^{(n-1)} d_{\rm f}^{(n-1)}}. \end{aligned}\right\} \dots\dots\dots\dots\dots \quad (18.6)$$

Putting

$$\begin{aligned} e_{\rm i} &= -f_{\rm i}\tan\alpha - k_{\rm i}\frac{\tan\gamma}{\cos\alpha} - g_{\rm i}, \\ e_{\rm i}' &= e_{\rm i} - x_{\rm i}^{n-1}\left(v_{\rm F}k_{\rm F}^{(n-2)} + v_{\rm G}k_{\rm G}^{(n-2)} + v_{\rm K}k_{\rm K}^{(n-2)}\right), \\ e_{\rm i}'' &= e_{\rm i}' - x_{\rm i}^{n-2}\left(v_{\rm F}k_{\rm F}^{(n-3)} + v_{\rm G}k_{\rm G}^{(n-3)} + v_{\rm K}k_{\rm K}^{(n-3)}\right), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ e_{\rm i}^{(n-2)} &= e_{\rm i}^{(n-3)} - x_{\rm i}^{2}\left(v_{\rm F}k_{\rm F}' + v_{\rm G}k_{\rm G}' + v_{\rm K}k_{\rm K}'\right), \end{aligned}$$

we may successively compute

$$\left.\begin{aligned} v_{\rm F}k_{\rm F}^{(n-2)} + v_{\rm G}k_{\rm G}^{(n-2)} + v_{\rm K}k_{\rm K}^{(n-2)} &= \frac{c_{\rm e}^{(n-2)}}{c^{(n-2)}}, \\ v_{\rm F}k_{\rm F}^{(n-3)} + v_{\rm G}k_{\rm G}^{(n-3)} + v_{\rm K}k_{\rm K}^{(n-3)} &= \frac{c_{\rm e'}^{(n-3)}}{c^{(n-3)}}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ v_{\rm F}k_{\rm F} + v_{\rm G}k_{\rm G} + v_{\rm K}k_{\rm K} &= \frac{c_{{\rm e}^{(n-2)}}^{(0)}}{c^{(0)}} = \frac{\left|1 \;\; e_{\rm i}^{(n-2)}\right|}{\left|1 \;\; x_{\rm i}\right|}, \\ v_{\rm F}\Delta F + v_{\rm G}\Delta G + v_{\rm K}\Delta K &= e_{\rm i}^{(n-2)} - x_{\rm i}\left(v_{\rm F}k_{\rm F} + v_{\rm G}k_{\rm G} + v_{\rm K}k_{\rm K}\right). \end{aligned}\right\} \dots\dots \quad (18.7)$$

After the computation of the angles α and γ according to the formula (*18.6*) and of the n expressions above, we have reduced the calculation problem in solving t_0, x_0, $\varepsilon M_{//}$, $\varepsilon M_{\perp}$ and the 2 n parameters of the remainder fields $F_{\rm r}$ and $G_{\rm r}$, from the data of two components $F_{\rm r}$ and $G_{\rm r}$ in $(n + 2)$ points. This problem is already treated in § 17.

§ 19. Calculation of actual determinants.

In the previous treatment of the calculation technique, determinants of the forms $c_F^{(m)}$, $c_{xPF}^{(m)}$ and $c^{(m)}$ dominate in the formulae. The key of calculation of these determinants is the calculation of the subdeterminant $c_{j(n+2)}^{(m)}$. As this determinant is a function of the distances $(x_i - x_q)$ (comp. *15.1*) between the datum points at the profile line, we may once for all compute the subdeterminants mentioned corresponding to a lot of systems of calculation points (point pattern) useful in interpretation work.

For the simple point pattern, where the distance $(x_i - x_q)$ between consecutive points has a constant value r we have according to (*15.1*)

$$\left.\begin{aligned} c_{1(n+2)}^{(n)} &= (-1)^{n+1} n!(n-1)!(n-2)! \ldots 2\cdot 1 \cdot r^{n\frac{(n+1)}{2}}, = \\ &= (-1)^{n+1}(n!)!\ r^{n\frac{(n+1)}{2}}, \\ c_{j(n+2)}^{(n)} &= (-1)^{j-1} \frac{(n+1)\,n\,(n-1)\ \ldots\ (n+3-j)}{(j-1)!}\, c_{1(n+2)}^{(n)}. \end{aligned}\right\} \ldots \quad (19.1)$$

If the determinants $c_F^{(n)}$, $c_{xF}^{(\ )}$ and $c^{(n)}$ have the indices $i = 1$ in the first row, $i = 2$ in the second row and so on all the way to $i = n + 2$ in the last row, we may write

$$r = x_2 - x_1 = x_3 - x_2 = \ldots = x_{n+2} - x_{n+1} \text{ and}$$

$$\left.\begin{aligned} c_F^{(n)} &= (-1)^{n+1}(n!)!\, r^{n\frac{(n+1)}{2}}\left[F_1 - \frac{n+1}{2}F_2 + \frac{(n+1)n}{1\cdot 2}F_3 - \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}F_4 \right. \\ &\quad \left. \ldots (-1)^{n+1} F_{n+2}\right], \\ c_{xF}^{(n)} &= (-1)^{n+1}(n!)!\ r^{n\frac{(n+1)}{2}}\left[x_1F_1 - \frac{n+1}{1}x_2F_2 + \frac{(n+1)n}{1\cdot 2}x_3F_3 \ \ldots \right. \\ &\quad \left. \ldots\ (-1)^{n+1} x_{n+2}F_{n+2}\right], \\ c^{(n)} &= (n+1)!n!\,(n-1)! \ldots 2\cdot 1\cdot r^{\frac{(n+1)(n+2)}{2}} = [(n+1)!]!\ r^{\frac{(n+1)(n+2)}{2}}. \end{aligned}\right\} \ldots \quad (19.2)$$

It is of interest to state that the coefficients of $F_1, F_2 \ldots F_{n+2}$ within the brackets are identical with the coefficients obtained by expanding a binome of the potence $(n + 1)$.

In the following, we demonstrate the way of calculation using the above mentioned point pattern in the case of one given component (F). Thereby we indicate the $(n + 4)$ given data of x_i, F_i with the suffix $i = 1, 2 \ldots (n + 4)$ and put $x_2 - x_1 = x_3 - x_2 = \ldots x_{n+4} - x_{n+3} = r$.

We let $c_F^{(n+1)}$ contain the rows from 1 to $(n+3)$ and $d_F^{(n+1)}$ the rows from 2 to $(n+4)$. Then

$$\left.\begin{aligned} c^{(n)} &= [(n+1)!]!\, r^{\frac{(n+1)(n+2)}{2}}, \\ c_F^{(n+1)} &= (-1)^{n+2} c^{(n)} \Big[F_1 - \frac{n+2}{1} F_2 \\ &\qquad + \frac{(n+2)(n+1)}{1 \cdot 2} F_3 \ldots (-1)^{n+2} F_{n+3} \Big], \\ d_F^{(n+1)} &= (-1)^{n+2} c^{(n)} \Big[F_2 - \frac{n+2}{1} F_3 \\ &\qquad + \frac{(n+2)(n+1)}{1 \cdot 2} F_4 \ldots (-1)^{n+2} F_{n+4} \Big]. \end{aligned}\right\} \quad (19.3)$$

The quantities, $c_{xF}^{(n+1)}$, $c_{x^2F}^{(n+1)}$, $d_{xF}^{(n+1)}$, $d_{x^2F}^{(n+1)}$, are obtained if we change $F_1, F_2 \ldots F_{(n+4)}$ in the above expressions in $x_1F_1, x_2F_2, \ldots x_{n+4}F_{n+4}$ and $x_1^2F_1, x_2^2F_2 \ldots x_{n+4}^2 F_{n+4}$ respectively. Thus we may easily compute the quantities contained in the formulae (*16.9*) for b_0 and b_1.

Let the determinant $c_F^{(n)}$ contain the rows from $i = 2$ to $i = (n+3)$, the determinant $c_{F'}^{(n-1)}$ the rows from $i = 2$ to $i = (n+2)$, the determinant $c_{F''}^{(n-2)}$ the rows from $i = 3$ to $i = (n+2)$ and so on all the way to $c_{F(n-1)}^{(1)}$ then we may write

$$\left.\begin{aligned} c^{(n-1)} &= (n!)!\, r^{\frac{n(n+1)}{2}},\quad c^{(n-2)} = [(n-1)!]!\, r^{\frac{(n-1)n}{2}} \text{ and so on, and} \\ c_F^{(n)} &= (-1)^{n+1} c^{(n-1)} \Big[F_2 - \frac{n+1}{1} F_3 + \frac{(n+1)n}{1 \cdot 2} F_4 \ldots (-1)^{n+1} F_{n+3} \Big], \\ c_{F'}^{(n-1)} &= (-1)^{n} c^{(n-2)} \Big[F_2' - \frac{n}{1} F_3' + \frac{n(n-1)}{1 \cdot 2} F_4' \ldots (-1)^{n} F_{n+2}' \Big], \\ c_{F''}^{(n-2)} &= (-1)^{n-1} c^{(n-3)} \Big[F_3'' - \frac{n-1}{1} F_4'' \\ &\qquad + \frac{(n-1)(n-2)}{1 \cdot 2} F_5'' \ldots (-1)^{(n-1)} F_{n+2}'' \Big], \end{aligned}\right\} \quad (19.4)$$

and so on all the way to $c_{F(n-1)}^{(1)}$.

These expressions facilitate to a high degree the computation of the quantities included in the formulae (*16.9*—*16.11*, *17.3*—*17.6*, *18.6*—*18.7*).

CHAPTER V.

Two Sheets.

§ 20. Interpretation equations.

The interpretation methods given in chapter II and III can easily be extended to the case, where the magnetic body may be considered as consisting of two sheets.

When the anomaly field F is the sum of the fields F_1 and F_2 from two sheets, we can according to (*12.2*) write

$$F = F_1\infty + F_2\infty + F_{r_1} + F_{r_2}.$$

Putting

$$F_r = F_{r_1} + F_{r_2} = \Delta F + k_F x + k_F' x^2 + \; . \; . \; . + k_F^{(n-2)} x^{n-1},$$

and marking the parameter functions a_0, a_1, b_0, b_1 with a second index (1) for one of the sheets and with a second index (2) for the other sheet, we have

$$F = \frac{a_{01} + a_{11}x}{-b_{01} - b_{11}x + x^2} + \frac{a_{02} + a_{12}x}{-b_{02} - b_{12}x + x^2} + \Delta F + k_F x + \ldots + k_F^{(n-2)} x^{n-1}.$$

This equation can be transformed into the form

$$A_0 + A_1 x + A_2 x^2 + \ldots + A_{n+3} x^{n+3} + B_0 F + B_1 xF + B_2 x^2 F + B_3 x^3 F = x^4 F, \; \ldots . \quad (20.1)$$

where

$$\left.\begin{aligned} A_0 &= A_{01} - B_0 \Delta F, \\ A_1 &= A_{11} - B_1 \Delta F - B_0 k_F \\ A_2 &= A_{21} - B_2 \Delta F - B_1 k_F - B_0 k_F', \\ A_3 &= A_{31} - B_3 \Delta F - B_2 k_F - B_1 k_F' - B_0 k_F'', \\ A_4 &= \Delta F - B_3 k_F - B_2 k_F' - B_1 k_F'' - B_0 k_F''', \\ A_5 &= k_F - B_3 k_F' - B_2 k_F'' - B_1 k_F''' - B_0 k_F'''', \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n+2} &= k_F^{(n-3)} - B_3 k_F^{(n-2)}, \\ A_{n+3} &= k_F^{(n-2)}, \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (20.2)$$

and

$$\left.\begin{aligned} A_{01} &= - a_{01}b_{02} - a_{02}b_{01}, \\ A_{11} &= - a_{01}b_{12} - a_{02}b_{11} - a_{11}b_{02} - a_{12}b_{01}, \\ A_{21} &= a_{01} + a_{02} - a_{11}b_{12} - a_{12}b_{11}, \\ A_{31} &= a_{11} + a_{12}, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (20.3)$$

and

$$\left.\begin{aligned} B_0 &= - b_{01}b_{02}, \\ B_1 &= - b_{01}b_{12} - b_{02}b_{11}, \\ B_2 &= b_{01} + b_{02} - b_{11}b_{12}, \\ B_3 &= b_{11} + b_{12}. \end{aligned}\right\} \qquad (20.4)$$

Equation (*20.1*) is the general form of the interpretation equation of the case of two sheets. The number of parameters of this equation is $(n + 8)$.

The procedure of interpretation calculations, based on equation (*20.1*), is in principle the following:

1) Determination of the parameters $A_0, A_1 \ldots A_{n+3}$ and B_0, B_1, B_2, B_3 from a system of interpretation equations, corresponding to $(n + 8)$ datum points at a reference line.

2) Computing the quantities $A_{01}, A_{11}, A_{21}, A_{31}$ and the coefficients ΔF, $k_F \ldots k_F^{(n-2)}$ of the potence series F_r. This can be made with the aid of the relations (*20.2*) or by carrying out the division $\dfrac{A_{n+3}x^{n+3} + A_{n+2}x^{n+2} \ldots A_0}{x^4 - B_3x^3 - B_2x^2 - B_1x - B_0}$.

3) Calculation of $b_{01}, b_{11}, b_{02}, b_{12}$ from the relations (*20.4*) and after that determination of a_{01}, a_{11} and a_{02}, a_{12} with the aid of the relations (*20.3*).

4) Computing, according to the formulae (*7.7*), $x_{01}, t_{01}, (\varepsilon M_{//}')_1, (\varepsilon M_{\perp}')_1$, corresponding to sheet (1) and of $x_{02}, t_{02}, (\varepsilon M_{//}')_2, (\varepsilon M_{\perp}')_2$, corresponding to sheet (2). Further estimation of the magnitudes of $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ of the sheets (1) and (2), based on the apparent values of these quantities which correspond to infinite length of the sheets.

5) For each of the two sheets: calculations from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ according to the description in § 8.

From the steps of calculation given above it is obviously only necessary to treat those pointed out in 1—3, since the computations according to points 4—5 are identical with those studied in chapters II and III.

Before the treatment of the calculation technique concerning the mentioned points 1—3, it is, however, convenient to consider the interpretation problem from a more general point of view.

INTERPRETATION EQUATION IN THE CASE OF (*m*) SHEETS

The interpretation equation corresponding to a number of (*m*) sheets can obviously be derived in a similar way as equation (*20.1*). In this case we have

$$F = \sum_{k=1}^{k=m} \frac{a_{ok} + a_{1k}x}{- b_{ok} - b_{1k}x + x^2} + F_r \qquad (20.5)$$

Putting

$$\varphi_p(x) = A_0 + A_1x + A_2x^2 + \ldots + A_px^p,$$
$$\Phi_p(x) = - B_0 - B_1x - B_2x^2 - \ldots - B_{p-1}x^{p-1} + x^p,$$

the interpretation equation, corresponding to (m) sheets and a remainder field F_r with (n) terms, can be written in the condensed form

$$\varphi_{(n+2m-1)}(x) - F \cdot \Phi_{2m}(x) = 0. \qquad (20.6)$$

Let us first study the above equation with regard to the stipulations that it has to correspond to an anomalous field derived from *real* sheets.

We assume that $\varphi_{(n+2m-1)}(x)$ and $\Phi_{2m}(x)$ (often designated in the following as $\varphi_{(n+2m-1)}$ and Φ_{2m}) have no common root and hence no common factor. Further, it may be stated that the problem mentioned above obviously implies a study of the conditions that allow the expression

$$F = \frac{\varphi_{(n+2m-1)}}{\Phi_{2m}}$$

to be split up into a potence series F_r, containing (n) terms, and into partial fractions each representing the field from a *real* sheet.

Since all coefficients entering in F_r and the partial fractions must be *real*, the same must also hold for the coefficients of $\varphi_{(n+2m-1)}$ and Φ_{2m}, *i. e.* all parameters of the equation (*20.5*) must be *real*.

According to (*20.5*) the denominators of the partial fractions can not include a higher potence of x than the second one. If the denominator of a partial fraction has a double root, it must be real, since the coefficients of the denominator are real. Further, it is easy to prove that the partial fraction in the mentioned case corresponds to a real sheet only if the numerator has the same root, *i. e.* if the partial fraction may be reduced to a simple fraction

$$\frac{a_1}{x - \frac{b_1}{2}}.$$

From the facts mentioned above and the known rules for splitting a fraction into simple partial fractions, it may be deduced that Φ_{2m}, in the case of real sheets, can only have *real single roots* and *conjugate complex single roots*. To each of the real single roots corresponds a sheet for which $t_0 = 0$. Every pair of conjugate complex roots correspond to a sheet for which t_0 is $\neq 0$.

Regarding the interpretation calculation it is according to the preceding natural to divide them into two head groups, viz. a) determination of the parameters of the interpretation equation and b) splitting up the fraction $\varphi_{(n+2m-1)} : \Phi_{2m}$ into a remainder field F_r and partial fractions. In the following, we treat partly some general principles for these calculations, partly the calculation technique in the case of two sheets.

§ 21. Determination of the parameters of the interpretation equation.

The parameters of $\varphi_{(n+2m-1)}(x)$ and $\Phi_{2m}(x)$ in equation (*20.6*) vary with the position of the zero point ($x = 0$) at the reference line. By a displacement of the zero point to the point $x = d$ at the reference line, the values A'_0, $A_1' \ldots A'_{(n+2m-1)}$ and B_0', $B_1' \ldots B'_{2m-1}$ of the parameters, corresponding to this new zero point, can conveniently be calculated according to Horner's method, well-known from the theory of equations.

Marking the successive derivates of $\varphi_\nu(x)$ and $\Phi_\nu(x)$ with one, two, three . . . dashes ('), ('') . . . $^{(\kappa)}$ we have

$$A_k' = \frac{\varphi^{(k)}_{(n+2m-1)}(d)}{k!}, \qquad B_k' = \frac{-\Phi^{(k)}_{2m}(d)}{k!}$$

and thus

$$A'_{(n+2m-1)} = A_{(n+2m-1)}, \qquad B'_{2m-1} = B_{2m-1} - 2md.$$

The parameters of $\varphi_{(n+2m-1)}(x)$ and $\Phi_{2m}(x)$, obviously, always can be calculated from a system of $(n + 4m)$ linear equations (*20.6*), corresponding to $(n + 4m)$ datum points at the reference line. Writing the head determinant of this equation system

$$D = \left| \, 1 \;\; x_i \;\; x_i^2 \; \ldots . \; x_i^{n+2m-1} \;\; F_i \;\; x_i F_i \; \ldots . \; x_i^{2m-1} F_i \, \right|$$

and using the symbols D_{A_k} and D_{B_k} for the determinants obtained, if the $(k + 1)$:th or the $(n + 2m + k + 1)$:th element in the above determinant of D is replaced by $x_i^{2m} F_i$, we have

$$A_k = D_{A_k} : D, \text{ and } B_k = D_{B_k} : D.$$

One or more of the equations in the system mentioned above may in certain circumstances be replaced by equations of another form. For instance if the F-curve has an extreme point $(x_e F_e)$ in one of the datum points, we obviously have

$$\varphi'_{(n+2m-1)}(x_e) - F_e \cdot \Phi'_{2m}(x_e) = 0.$$

Should also the second derivative at the extreme point be known we obtain yet another equation, viz.

$$\varphi''_{(n+2m-1)}(x_e) - F_e \cdot \Phi''_{2m}(x_e) - F_e'' \cdot \Phi_{2m}(x_e) = 0.$$

where F_e'' denotes the second derivative of F with regard to x in the mentioned extreme point. Since the radius of curvature at an arbitrary point on the F-curve is generally given by

$$R = \frac{(1 + F')^{3/2}}{F''},$$

we have

$$F_e'' = \frac{1}{R_e},$$

where R_e is the radius of curvature in the extreme point. Making $x_e = 0$, we obtain from the three relations above

$$\left. \begin{aligned} & A_0 + F_e B_0 = 0, \\ & A_1 + F_e B_1 = 0, \\ & 2R_e(A_2 + F_e B_2) + B_0 = 0. \end{aligned} \right\} \qquad (21.1)$$

Let us regard a datum point (i) at which F and the first k derivatives of F are known. By repeatedly differentiating the interpretation equation (20.6) and by choosing $x_i = 0$, we obtain

$$A_0 + F_i B_0 = 0,$$
$$A_1 + F_i B_1 + F_i' B_0 = 0,$$
$$A_2 + F_i B_2 + F_i' B_1 + F_i'' B_0 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_k + F_i B_k + \frac{F_i'}{1} B_{k-1} + \frac{F_i''}{1 \cdot 2} B_{k-2} \cdots + \frac{F_i^{(k)}}{k!} B_0 = 0.$$

Hence it is possible, from a theoretical point of view, to calculate all parameters of the interpretation equation, if F and the first $(n + 4m - 1)$ derivatives of this quantity are known in a datum point. In practical interpretation work, however, it is generally not possible to obtain the mentioned derivatives of a higher order with an accuracy, satisfactory for the calculations, from an F-curve derived from field measurements. Even the second derivative will in most cases be determined with too great an uncertainty.

In the following we treat the calculation technique in the case of two sheets.

In the general case, *i. e.* by an arbitrary location of the datum points in the interpretation diagram (Fig. 5.1), the calculations become very tedious. In making special selections of datum points, however, the work of calculation may be much simplified. We first treat a number of such selections and presuppose at this treatment that the remainder field (F_r) can be omitted, *i. e.* the interpretation is

$$A_0 + A_1 x + A_2 x^2 + A_3 x^3 + B_0 F + B_1 x F + B_2 x^2 F + B_3 x^3 F = x^4 F.$$

Simple cases.

1) Given four datum points with $F = F_1$ and four datum points with $F = F_2$.

Since the interpretation equation is of the 4th degree with respect to x, a straight line ($F = F_1$) in the interpretation diagram will cut the F-curve in four points at the most. Denoting the x-coordinates of the cutting points by x_1, x_1', x_1'', x_1''', we obviously have

$$\varphi_3(x) - F_1 \Phi_4(x) = - F_1 (x - x_1)(x - x_1')(x - x_1'')(x - x_1''').$$

From this relation we obtain

$$\left.\begin{aligned} A_0 + F_1 B_0 &= - x_1 x_1' x_1'' x_1''' F_1 = - [xxxx]_1 F_1, \\ A_1 + F_1 B_1 &= (x_1 x_1' x_1'' + x_1 x_1' x_1''' + x_1 x_1'' x_1''' + x_1' x_1'' x_1''') F_1 \\ &= [xxx]_1 F_1, \\ A_2 + F_1 B_2 &= - (x_1 x_1' + x_1 x_1'' + x_1 x_1''' + x_1' x_1'' + x_1' x_1''' \\ &\quad + x_1'' x_1''') F_1 = - [xx]_1 F_1, \\ A_3 + F_1 B_3 &= (x_1 + x_1' + x_1'' + x_1''') F_1 = [x]_1 F_1. \end{aligned}\right\} \quad \ldots\ldots \quad (21.2)$$

Completely analogous equations hold for the datum points corresponding to a line $F = F_2$ in the interpretation diagram. Denoting the quantities containing abscissae terms by suffix (2) and solving A_0 and B_0 from the first of these equations and the first of the equations (*21.2*) and solving A_1 and B_1 from the following two equations and so on, we obtain

$$\left.\begin{aligned} A_0 &= \frac{F_1F_2([xxxx]_2 - [xxxx]_1)}{F_2 - F_1}, & B_0 &= -\frac{F_2[xxxx]_2 - F_1[xxxx]_1}{F_2 - F_1}, \\ A_1 &= -\frac{F_1F_2([xxx]_2 - [xxx]_1)}{F_2 - F_1}, & B_1 &= \frac{F_2[xxx]_2 - F_1[xxx]_1}{F_2 - F_1}, \\ A_2 &= \frac{F_1F_2([xx]_2 - [xx]_1)}{F_2 - F_1}, & B_2 &= -\frac{F_2[xx]_2 - F_1[xx]_1}{F_2 - F_1}, \\ A_3 &= -\frac{F_1F_2([x]_2 - [x]_1)}{F_2 - F_1}, & B_3 &= \frac{F_2[x]_2 - F_1[x]_1}{F_2 - F_1}. \end{aligned}\right\} \quad \ldots\ldots \quad (21.3)$$

2) GIVEN EIGHT DATUM POINTS OF WHICH FOUR HAVE THE SAME F-VALUES.

Let us denote the abscissae of the datum points, corresponding to $F = F_1$, by x_1, x_1', x_1'', x_1'''. Further we make use of the notations $[xxxx]_1$, $[xxx]_1$, $[xx]_1$, $[x]_1$ according to equation (*21.2*).

Since the relations (*21.2*) are also valid in this instance we may write

$$\left.\begin{aligned} B_0 &= -\frac{A_0}{F_1} - [xxxx]_1, & B_2 &= -\frac{A_2}{F_1} - [xx]_1, \\ B_1 &= -\frac{A_1}{F_1} + [xxx]_1, & B_3 &= -\frac{A_3}{F_1} + [x]_1. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (21.4)$$

If we insert these expressions for $B_0 \ldots B_3$ in the interpretation equation it may be written

$$A_0 + A_1x + A_2x^2 + A_3x^3 = \frac{F_1F}{F_1 - F}\Big[x^4 - x^3[x]_1 + x^2[xx]_1 - x[xxx]_1 + [xxxx]_1\Big]. \quad \ldots\ldots\ldots\ldots \quad (21.5)$$

On inserting the coordinates (x_iF_i) for four other datum points in this equation we obtain a system of four equations from which $A_0 \,..\, A_3$ may be solved. Thereupon the parameters $B_0 \,..\, B_3$ are easily calculated with the aid of expressions (*21.4*).

The calculations may be further simplified by suitably choosing the four last mentioned datum points (x_iF_i). In the following, two such cases are treated.

a) *Two of the other four datum points have a common value $F = F_2$.*

We denote the data corresponding to the four datum points by x_2F_2, $x_2'F_2$, x_3F_3, x_4F_4 and choose the origin on the x-axis so that $x_2 + x_2' = 0$.

On adding and subtracting the two equations (*21.5*), corresponding to the datum points with the common F value, we obtain

$$\left.\begin{aligned} A_0 + A_2 x_2^2 &= \frac{F_1 F_2}{F_1 - F_2}\left[x_2^4 + x_2^2 [xx]_1 + [xxxx]_1\right] = E_0 F_1, \\ A_1 + A_3 x_2^2 &= -\frac{F_1 F_2}{F_1 - F_2}\left[x_2^2 [x]_1 + [xxx]_1\right] = E_1 F_1. \end{aligned}\right\} \ldots\ldots\ldots\ldots (21.6)$$

On putting

$$C_i = \frac{F_i}{F_1 - F_i}\left(x_i^4 - x_i^3 [x]_1 + x_i^2 [xx]_1 - x_i [xxx]_1 + [xxxx]_1\right). \quad \ldots. \quad (21.7)$$

we obtain with the aid of equations (*21.6*) and two equations (*21.5*) corresponding to the datum points (3) and (4)

$$\left.\begin{aligned} A_0 &= E_0 F_1 - A_2 x_2^2, \\ A_1 &= E_1 F_1 - A_3 x_2^2, \\ A_2 &= -\frac{x_3(x_3^2 - x_2^2)(C_4 - E_0 - E_1 x_4) - x_4 (x_4^2 - x_2^2)(C_3 - E_0 - E_1 x_3)}{(x_4 - x_3)(x_4^2 - x_2^2)(x_3^2 - x_2^2)} F_1, \\ A_3 &= \frac{(x_3^2 - x_2^2)(C_4 - E_0 - E_1 x_4) - (x_4^2 - x_l^2)(C_3 - E_0 - E_1 x_3)}{(x_4 - x_3)(x_4^2 - x_2^2)(x_3^2 - x_2^2)} F_1 \end{aligned}\right\} (21.8)$$

Special case: Should the points $(x_2 F_2)$ and $(x_2' F_2)$ coincide with an extreme point $(x_e F_e)$ we have $x_2 = x_2' = 0$ and $F_2 = F_e$. Equations (*21.8*) may then be written

$$\left.\begin{aligned} A_0 &= \frac{F_1 F_e}{F_1 - F_e}[xxxx]_1, \\ A_1 &= \frac{F_1 F_e}{F_1 - F_e}[xxx]_1, \\ A_2 &= -\frac{x_3^3(C_4 F_1 - A_0 - A_1 x_4) - x_4^3(C_3 F_1 - A_0 - A_1 x_3)}{x_3^2 x_4^2 (x_4 - x_3)}, \\ A_3 &= \frac{x_3^2(C_4 F_1 - A_0 - A_1 x_4) - x_4^2(C_3 F_1 - A_0 - A_1 x_3)}{x_3^2 x_4^2 (x_4 - x_3)}. \end{aligned}\right\} \ldots\ldots\ldots\ldots (21.9)$$

b) *The four remaining datum points* (*2, 3, 4, 5*) *are chosen so that* $x_2 + x_5 = x_3 + x_4 = 0$.

The four equations (*21.5*) that correspond to the remaining datum points may, by addition and subtraction pairwise, be transformed to

$$A_0 + A_2 x_2^2 = \frac{C_2 + C_5}{2} F_1, \qquad A_1 x_2 + A_3 x_2^3 = \frac{C_2 - C_5}{2} F_1,$$

$$A_0 + A_2 x_3^2 = \frac{C_3 + C_4}{2} F_1, \qquad A_1 x_3 + A_3 x_3^3 = \frac{C_3 - C_4}{2} F_1.$$

Hence

$$\left.\begin{aligned} A_0 &= \frac{x_3^2(C_2+C_5)-x_2^2(C_3+C_4)}{2(x_3^2-x_2^2)}F_1, \quad A_2=-\frac{(C_2+C_5)-(C_3+C_4)}{2(x_3^2-x_2^2)}F_1. \\ A_1 &= \frac{x_3^3(C_2-C_5)-x_2^3(C_2-C_4)}{2x_2x_3(x_3^2-x_2^2)}F_1, \quad A_3=-\frac{x_3(C_2-C_5)-x_2(C_3-C_4)}{2x_2x_3(x_3^2-x_2^2)}F_1. \end{aligned}\right\} \quad (21.10)$$

3) Two of the eight datum points have a common value of F.

We denote the common F-value with F_1 and the corresponding x-values x_1 and x_1'. Included is also the case of $x_1 = x_1' = x_e$ and $F_1 = F_e$. Further let $\alpha_1 = x_1''x_1'''$ and $\beta_1 = x_1'' + x_1'''$ whereby x_1'' and x_1''' denote the two roots (real or complex) that, along with x_1 and x_1' satisfy the interpretation equation of $F = F_1$. According to (21.2) the following relationships then hold.

$$\left.\begin{aligned} [xxxx]_1 &= \alpha_1 x_1 x_1', \\ [xxx]_1 &= \alpha_1(x_1+x_1') + \beta_1 x_1 x_1', \\ [xx]_1 &= \alpha_1 + \beta_1(x_1+x_1') + x_1 x_1', \\ [x]_1 &= \beta_1 + x_1 + x_1'. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (21.11)$$

Solving $B_0 \ldots B_3$ from (21.2) and inserting these expressions in the interpretation equation, it can be written

$$\left.\begin{aligned} & A_0 + A_1x_i + A_2x_i^2 + A_3x_i^3 - \alpha_1 U_iF_1 + \beta_1 U_iF_1x_i = U_iF_1x_i^2, \\ & \text{where } U_i = \frac{F_i}{F_1 - F_i}(x_i - x_1)(x_i - x_1'). \end{aligned}\right\} \quad \ldots\ldots \quad (21.12)$$

Equation (21.12) can be applied to the other six datum points for calculation of α_1, β_1, $A_0 \ldots A_3$. By these calculations it is convenient to begin with the determination of α_1 and β_1.

For a system, containing five equations (21.12), we have the determinant

$$\left| 1 \;\; x_i \;\; x_i^2 \;\; x_i^3 \;\; (-\alpha_1U_iF_1 + \beta_1U_iF_1x_i - U_iF_1x_i^2) \right| = 0.$$

On using the symbols of determinants from § 15, this expression can be written

$$-\alpha_1 c_{\mathrm{U}}^{(3)} + \beta_1 c_{\mathrm{xU}}^{(3)} = c_{\mathrm{x^2U}}^{(3)},$$

$$\text{or } -\alpha_1 d_{\mathrm{U}}^{(3)} + \beta_1 d_{\mathrm{xU}}^{(3)} = d_{\mathrm{x^2U}}^{(3)}.$$

Hence

$$\left.\begin{aligned} \alpha_1 &= \frac{c_{\mathrm{xU}}^{(3)} d_{\mathrm{x^2U}}^{(3)} - c_{\mathrm{x^2U}}^{(3)} d_{\mathrm{xU}}^{(3)}}{c_{\mathrm{U}}^{(3)} d_{\mathrm{xU}}^{(3)} - c_{\mathrm{xU}}^{(3)} d_{\mathrm{U}}^{(3)}}, \\ \beta_1 &= \frac{c_{\mathrm{U}}^{(3)} d_{\mathrm{x^2U}}^{(3)} - c_{\mathrm{x^2U}}^{(3)} d_{\mathrm{U}}^{(3)}}{c_{\mathrm{U}}^{(3)} d_{\mathrm{xU}}^{(3)} - c_{\mathrm{xU}}^{(3)} d_{\mathrm{U}}^{(3)}}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (21.13)$$

After the determination of α_1 and β_1 it is often expedient to calculate

$$x_1'' = \tfrac{1}{2}\left[\beta_1 + \sqrt{-4\,\alpha_1 + \beta_1^2}\right] \text{ and } x_1''' = \tfrac{1}{2}\left[\beta_1 - \sqrt{-4\,\alpha_1 + \beta_1^2}\right]. \quad \ldots\ldots \quad (21.14)$$

If x_1'' and x_1''' are real, then the F-curve must contain the points (x_1'', F_1) and (x_1''', F_1) according to the interpretation equation. However, it may occur that such a course of the F-curve is in bad agreement with the measured F-curve, drawn in an interpretation diagram according to (Fig. 5.1). Such being the case, the calculations ought to be interrupted since they obviously do not give satisfactory results.

If x_1'' and x_1''' are complex numbers, then the interpretation equation generally corresponds to two sheets, having $t \neq 0$ (comp. § 22 pp. 81—83).

By determination of $A_0 \,..\, A_3$ we use four equations

$$\left.\begin{aligned} &A_0 + A_1 x_i + A_2 x_i^2 + A_3 x_i^3 = V_i F_1, \\ \text{where}\quad &V_i = (\alpha_1 - \beta_1 x_i + x_i^2)\, U_i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21.15)$$

By determination of $B_0 \,..\, B_3$ we calculate at first $[xxxx]_1$, $[xxx]_1$, $[xx]_1$, $[x]_1$ with the aid of the expressions (*21.11*) and obtain after that the desired quantities from (*21.4*).

By a suitable choice of the six datum points (2, 3 . . . 7) the calculations will be further simplified. In the following, two such cases are treated.

a) *Two of the other six datum points have the same value of F.*

We denote the common F-value with F_2 and the corresponding x-coordinates with x_2 and x_2'. Further we put $\alpha_2 = x_2'' x_2'''$ and $\beta_2 = x_2'' + x_2'''$, where x_2'' and x_2''' are roots of the interpretation equation when $F = F_2$.

In the same manner as in the preceding, we may derive expressions for $[xxxx]_2$, $[xxx]_2$, $[xx]_2$, $[x]_2$. These expressions proceed from (*21.11*) if suffix (1) is changed to suffix (2).

Further, on putting

$$U_i' = \frac{F_i}{F_2 - F_1}\,(x_i - x_2)\,(x_i - x_2'),$$

an interpretation equation, containing α_2, β_2, U_i', may be written analogously to that in (*21.12*).

On subtracting the two interpretation equations mentioned we obtain

$$\left.\begin{aligned} &\alpha_1 - \beta_1 x_i - \alpha_2 P_i + \beta_2 x_i P_i = x_i^2\,(P_i - 1), \\ \text{where}\quad &P_i = \frac{U_i' F_2}{U_i F_1} = \frac{F_2\,(F_1 - F_i)\,(x_i - x_2)\,(x_i - x_2')}{F_1\,(F_2 - F_i)\,(x_i - x_1)\,(x_i - x_1')}. \end{aligned}\right\} \quad \ldots\ldots \quad (21.16)$$

Thus, it is possible to solve α_1, β_1, α_2, β_2 by applying the above equation to the four remaining datum points (3 . . 6).

After the determination of these four quantities, we obtain $[xxxx]_1$, $[xxx]_1$ $[xx]_1$, $[x]_1$ from the equations (*21.11*) and in the same manner we calculate the analogous quantities, corresponding to suffix (2).

Finally formulae (*21.3*) yield the values of $A_0 \,..\, A_3$ and $B_0 \,..\, B_3$.

b) *The remaining six datum points* (2 . . . . 7) *have* $x_2 + x_7 = x_3 + x_6 = x_4 + x_5$.

First we calculate

$$\left.\begin{aligned} u_2 &= \tfrac{1}{2}(U_2 - U_7), & v_2 &= \tfrac{1}{2}(U_2 + U_7), \\ u_3 &= \tfrac{1}{2}(U_3 - U_6), & v_3 &= \tfrac{1}{2}(U_3 + U_6), \\ u_4 &= \tfrac{1}{2}(U_4 - U_5), & v_4 &= \tfrac{1}{2}(U_4 + U_5). \end{aligned}\right\} \quad \ldots\ldots \quad (21.17)$$

Assume that F-curves, corresponding to single sheets, are drawn through the points (x_1, F_1) and (x_1', F_1), and further through the two points (x_2, F_2) and (x_7, F_7), or the two points (x_3, F_3) and (x_6, F_6), or the two points (x_4, F_4) and (x_5, F_5). It can easily be proved that the parameters (a_0, a_1, b_0, b_1), corresponding to the three mentioned F-curves, are obtained from the formulae

$$\left.\begin{aligned} a_{02} &= v_2 F_1, & a_{12} &= \frac{u_2}{x_2} F_1, & b_{02} &= -v_2 - x_1 x_1', & b_{12} &= -\frac{u_2}{x_2} + x_1 + x_1', \\ a_{03} &= v_3 F_1, & a_{13} &= \frac{u_3}{x_3} F_1, & b_{03} &= -v_3 - x_1 x_1', & b_{13} &= -\frac{u_3}{x_3} + x_1 + x_1', \\ a_{04} &= v_4 F_1, & a_{14} &= \frac{u_4}{x_4} F_1, & b_{04} &= -v_4 - x_1 x_1', & b_{14} &= -\frac{u_4}{x_4} + x_1 + x_1', \end{aligned}\right\} \quad (21.18)$$

where the parameters are marked with a second suffix for separating the different curves.

If $v_2 = v_3 = v_4$ and $\frac{u_2}{x_2} = \frac{u_3}{x_3} = \frac{u_4}{x_4}$ then the above parameters are identical and the three F-curves coincide. It obviously implies that the F-curve in the interpretation diagram emanates from one sheet, and the magnetic parameters of this sheet can be calculated from the mentioned parameter functions.

By practical interpretation work the actual F-curve can often be considered as emanating from one sheet even if the three groups of parameter functions mentioned above have a little deviating values. When the agreement is not satisfactory, the calculations are continued in the following manner.

On calculating

$$\left.\begin{aligned} c_1 &= u_2 \frac{x_3^2 - x_4^2}{x_2} - u_3 \frac{x_2^2 - x_4^2}{x_3} + u_4 \frac{x_2^2 - x_3^2}{x_4}, \\ c_2 &= v_2 (x_3^2 - x_4^2) - v_3 (x_2^2 - x_4^2) + v_4 (x_2^2 - x_3^2), \\ c_3 &= u_2 x_2 (x_3^2 - x_4^2) - u_3 x_3 (x_2^2 - x_4^2) + u_4 x_4 (x_2^2 - x_3^2), \\ c_4 &= v_2 x_2^2 (x_3^2 - x_4^2) - v_3 x_3^2 (x_2^2 - x_4^2) + v_4 x_4^2 (x_2^2 - x_3^2), \end{aligned}\right\} \quad \ldots\ldots \quad (21.19)$$

we may obtain α_1 and β_1 from the expressions

$$\alpha_1 = \frac{c_3^2 - c_2 c_4}{c_2^2 - c_1 c_3}, \qquad \beta_1 = \frac{c_2 c_3 - c_1 c_4}{c_2^2 - c_1 c_3}. \quad \ldots\ldots\ldots\ldots \quad (21.20)$$

After the computation of the above quantities it is often convenient to determine x_1'' and x_1''' according to (*21.14*) and with the aid of these values judge the probability of satisfactory results by a completing of the calculations.

It is now possible to compute $[xxxx]_1$, $[xxx]_1$, $[xx]_1$, $[x]_1$ from the formulae (*21.11*). Further we calculate

$$\left.\begin{array}{ll} e_2 = v_2 (x_2^2 + \alpha_1) - u_2 \beta_1 x_2, & g_2 = u_2 (x_2^2 + \alpha_1) - v_2 \beta_1 x_2, \\ e_3 = v_3 (x_3^2 + \alpha_1) - u_3 \beta_1 x_3, & g_3 = u_3 (x_3^2 + \alpha_1) - v_3 \beta_1 x_3, \\ e_4 = v_4 (x_4^2 + \alpha_1) - u_4 \beta_1 x_4, & g_4 = u_4 (x_4^2 + \alpha_1) - v_4 \beta_1 x_4. \end{array}\right\} \ldots\ldots \quad (21.21)$$

Then

$$\left.\begin{array}{l} A_0 = \dfrac{e_2 x_3^2 - e_3 x_2^2}{x_3^2 - x_2^2} F_1 = \dfrac{e_2 x_4^2 - e_4 x_2^2}{x_4^2 - x_2^2} F_1 = \dfrac{e_3 x_4^2 - e_4 x_3^2}{x_4^2 - x_3^2} F_1, \\ A_1 = \dfrac{g_2 x_3^3 - g_3 x_2^3}{x_2 x_3 (x_3^2 - x_2^2)} F_1 = \dfrac{g_2 x_4^3 - g_4 x_2^3}{x_2 x_4 (x_4^2 - x_2^2)} F_1 = \dfrac{g_3 x_4^3 - g_4 x_3^3}{x_3 x_4 (x_4^2 - x_3^2)} F_1, \\ A_2 = \dfrac{e_3 - e_2}{x_3^2 - x_2^2} F_1 = \dfrac{e_4 - e_2}{x_4^2 - x_2^2} F_1 = \dfrac{e_4 - e_3}{x_4^2 - x_3^2} F_1, \\ A_3 = \dfrac{g_3 x_2 - g_2 x_3}{x_2 x_3 (x_3^2 - x_2^2)} F_1 = \dfrac{g_4 x_2 - g_2 x_4}{x_2 x_4 (x_4^2 - x_2^2)} F_1 = \dfrac{g_4 x_3 - g_3 x_4}{x_3 x_4 (x_4^2 - x_3^2)} F_1. \end{array}\right\} \ldots \quad (21.22)$$

The parameters $B_0 \ldots B_3$ may be calculated from the formulae (*21.4*).

General case.

The interpretation equation

$$\varphi_{(n+3)}(x) - F \cdot \Phi_4(x) = 0$$

has $(n + 8)$ parameters. Thus, if $(n + 8)$ datum points are known, it is possible to determine the parameters by calculating $(n + 9)$ determinants, $D_{A_0}, \ldots D_{A_{n+3}}, D_{B_0} \ldots D_{B_3}$ and D (comp. p. 71) of the equation system, corresponding to the mentioned datum points. It is often more convenient, however, to split up the calculations in the following manner.

For a system of $(n + 5)$ equations we have the determinant

$$\left| 1 \;\; x_i \;\; x_i^2 \ldots \; x_i^{n+3} \;\; F_i(B_0 + B_1 x_i + B_2 x_i^2 + B_3 x_i^3 - x_i^4) \right| = 0.$$

On using the symbols of determinants from § 15, this expression can be written

$$B_0 c_{F}^{(n+3)} + B_1 c_{xF}^{(n+3)} + B_2 c_{x^2F}^{(n+3)} + B_3 c_{x^3F}^{(n+3)} = c_{x^4F}^{(n+3)}. \ldots\ldots\ldots\ldots \quad (21.23)$$

Let the above equation correspond to the datum points $(1, 2 \ldots (n + 5))$. Then equations of analogous form correspond to the datum points $(2, 3 \ldots (n + 6))$, or the datum points $(3, 4 \ldots (n + 7))$, or the datum points $(4, 5 \ldots (n + 8))$. The determinants in these equations are obviously different and may be denoted with the symbols, $d_F^{(n+3)} \ldots, e_F^{(n+3)} \ldots, f_F^{(n+3)} \ldots$ From the four mentioned equations we calculate $B_0 \ldots B_3$.

After the determination of the parameters $B_0 \ldots B_3$ it is convenient to carry out the calculations of b_{01}, b_{11} and b_{02}, b_{12} according to the method given in the next paragraph. By this proceeding it can be established whether the actual interpretation trial gives real results before the beginning of the calculation of $A_0 \ldots A_{n+3}$. Further, it is often not necessary to determine the last mentioned parameters by solving the actual interpretation problem, for instance by the determination of the depth of an ore body.

When it is desirable to calculate the parameters $A_0 \ldots A_{n+3}$ it can be made from $(n+4)$ equations of the form

$$\left.\begin{aligned} &A_0 + A_1 x_i + A_2 x_i^2 \ldots + A_{n+3} x_i^{n+3} = Q_i F_i, \\ \text{where} \quad &Q_i = (x_i^4 - B_3 x_i^3 - B_2 x_i^2 - B_1 x_i - B_0)\, F_i. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (21.24)$$

§ 22. Determination of the parameters of the single sheets.

After the calculation of the parameters of an interpretation equation we have to split up the fraction $\varphi_{(n+2m-1)}(x) : \Phi_{2n}(x)$ into a remainder field (F_r) and partial fractions corresponding to single sheets.

By arranging the terms of the polynoms $\varphi_{(n+2m-1)}(x)$ and $\Phi_{2m}(x)$ according to descending powers of x and dividing the former polynom with the latter we obtain

$$F = \frac{\varphi_{(n+2n-1)}(x)}{\Phi_{2m}(x)} = k_F^{(n-2)} x^{n-1} + \ldots + k_F + \Delta F$$

$$+ \frac{A_{(2m-1)} x^{2m-1} + \ldots + A_1 x + A_0}{x^{2m} - B_{2m-1} x^{2m-1} \ldots - B_1 x - B_0} = F_r + \frac{\varphi_{(2m-1)}(x)}{\Phi_{2m}(x)}.$$

In § 20 it has been pointed out that a proper fraction $\varphi_{2m-1} \div \Phi_{2m}$ can be splitted up in partial fractions corresponding to only real sheets, if Φ_{2m} has only single roots. Thus the first criterion of a satisfactory result is that Φ_{2m} and its derivative Φ'_{2m} have no common root, *i. e.* no common factor. The procedure of investigating this state is identical with the well-known method of calculating the highest common factor for two polynoms.

In an interpretation equation that represents the field correctly along the whole length of the reference line (and not only in the chosen datum points), the polynomial Φ_{2m} can not generally have any real roots. The reasons for this fact are treated later on in this section. Thus the second criterion of satisfactory interpretation results is in almost all actual cases that Φ_{2m} has only complex roots. By the aid of the well-known Sturm's theorem, it is possible to state how many of the roots of Φ_{2m} are real and how many are complex.

In separating $\varphi_{2m-1} : \Phi_{2m}$ into partial fractions according to (*20.5*) it is necessary to know the roots of Φ_{2m} or, what is in principle the same thing, to determine the coefficients included in the factors $(x^2 - b_{1k}x - b_{0k})$. When Φ_{2m} is of a higher degree than the fourth, these calculations must be made with the aid of methods of approximation given in the theory of equations.

After the factorizing of Φ_{2m} the numerators of the partial fractions can be determined according to the method of undetermined coefficients.

In the following, we treat the *calculation technique in the case of two sheets.* Thereby we write

$$\frac{\varphi_3(x)}{\Phi_4(x)} = \frac{A_0 + A_1x + A_2x^2 + A_3x^3}{-B_0 - B_1x - B_2x^2 - B_3x^3 + x^4},$$

where the parameters $A_0 \,..\, A_3$ are identical with the symbols $A_{01} \,..\, A_{31}$ in equations (*20.3*) and the formulae (*20.4*) are valid for the parameters $B_0 \ldots B_3$.

From the expressions for B_1 and B_3 in (*20.4*) we obtain

$$b_{11} = \frac{B_1 + b_{01}B_3}{b_{01} - b_{02}}, \qquad b_{12} = -\frac{B_1 + b_{02}B_3}{b_{01} - b_{02}}. \quad \ldots\ldots\ldots\ldots\ldots \quad (22.1)$$

Thus

$$-b_{11}b_{12} = \frac{(B_1 + b_{01}B_3)\,(B_1 + b_{02}B_3)}{(b_{01} - b_{02})^2} = \frac{B_1^2 + B_1B_3(b_{01} + b_{02}) - B_0B_3^2}{(b_{01} + b_{02})^2 + 4B_0}$$

and

$$B_2 = (b_{01} + b_{02}) + \frac{B_1^2 + B_1B_3(b_{01} + b_{02}) - B_0B_3^2}{(b_{01} + b_{02})^2 + 4B_0}.$$

Hence

$$(b_{01} + b_{02})^3 - B_2(b_{01} + b_{02})^2 + (4B_0 + B_1B_3)\,(b_{01} + b_{02}) = 4B_0B_2 + B_0B_3^2 - B_1^2. \quad \ldots. \quad (22.2)$$

Further we obviously may put

$$\left. \begin{aligned} b_{01} &= \tfrac{1}{2}\left[(b_{01} + b_{02}) + \sqrt{4B_0 + (b_{01} + b_{02})^2}\right], \\ b_{02} &= \tfrac{1}{2}\left[(b_{01} + b_{02}) - \sqrt{4B_0 + (b_{01} + b_{02})^2}\right]. \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots \quad (22.3)$$

With the aid of equations (*22.1, 22.2, 22.3*) it is possible to determine b_{01}, b_{02} and b_{11}, b_{12}. For this we first calculate $(b_{01} + b_{02})$ from (*22.2*) then b_{01} and b_{02} from (*22.3*) and finally b_{11} and b_{12} from (*22.1*). When $b_{01} = b_{02}$ formulae (*22.1*) assume the indefinite form 0 : 0. In this case the following formulae hold

$$\left. \begin{aligned} b_{11} &= \tfrac{1}{2}\left[B_3 + \sqrt{B_3^2 + 4\{B_2 - (b_{01} + b_{02},)\}}\ \right], \\ b_{12} &= \tfrac{1}{2}\left[B_3 - \sqrt{B_3^2 + 4\{B_2 - (b_{01} + b_{02})\}}\ \right]. \end{aligned} \right\} \quad \ldots\ldots\ldots\ldots \quad (22.4)$$

When the parameter functions above mentioned have been determined, we may calculate a_{01}, a_{02} and a_{11}, a_{12} from equations (*20.3*).

If we put

$$\left.\begin{aligned}
D &= (b_{01} - b_{02})^2 + (b_{11} - b_{12})(b_{01}b_{12} - b_{02}b_{11}),\\
D_{a_{01}} &= (b_{01} - b_{02})(A_0 + b_{01}A_2) + (b_{11} - b_{12})(b_{11}A_0 - b_{01}A_1)\\
&\qquad + b_{01}A_3(b_{01}b_{12} - b_{02}b_{11}),\\
D_{a_{11}} &= (b_{01} - b_{02})(A_1 + b_{01}A_3) - (b_{11} - b_{12})A_0\\
&\qquad + (b_{01}b_{12} - b_{02}b_{11})(A_2 + b_{11}A_3),\\
&\text{then}\\
a_{01} &= D_{a_{01}} : D, \qquad a_{11} = D_{a_{11}} : D,\\
a_{02} &= -\frac{A_0 + a_{01}b_{02}}{b_{01}}, \quad a_{12} = A_3 - a_{11}.
\end{aligned}\right\} \quad \ldots\ldots \quad (22.5)$$

On calculating $(b_{01} + b_{02})$ from equation (*22.2*) it is better first to compute

$$\left.\begin{aligned}
p &= 4B_0 + B_1B_3 - \frac{1}{3}B_2^2,\\
q &= B_1^2 - B_0B_3^2 + B_2\left(\frac{p}{3} - 4B_0 + \frac{B_2^2}{27}\right).
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22.6)$$

Equation (*22.2*) then may be written

$$\left[(b_{01} + b_{02}) - \frac{B_2}{3}\right]^3 + p\left[(b_{01} + b_{02}) - \frac{B_2}{3}\right] + q = 0. \quad \ldots\ldots\ldots\ldots \quad (22.7)$$

The character of the values of $(b_{01} + b_{02})$, which satisfy the equation (*22.7*) and the character of the values of x, which satisfy the polynom $\Phi_4(x)$ is determined by the quantity

$$Q = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22.8)$$

The following cases can be separated:

a) $Q > 0$.

Equation (*22.7*) has one real and two conjugate complex roots. The real root is

$$\left.\begin{aligned}
(b_{01} + b_{02}) &= \frac{B_2}{3} + u - \frac{p}{3u},\\
\text{where} \qquad u &= \left[-\frac{q}{2} + \sqrt{Q}\right]^{1/3}.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22.9)$$

$\Phi_4(x)$ has two real and two conjugate complex roots.

The interpretation equation (*20.1*) generally represents a field F, emanating from one sheet with $t_0 \neq 0$ and two sheets with $t_0 = 0$.

If $\Phi_4(x)$ and $\varphi_3(x)$ have a common factor, the number of sheets is reduced. Is this factor of the second degree in x, the interpretation equation is also of the second degree and gives the field F from one sheet with $t_0 \neq 0$ or from two sheets with $t_0 = 0$.

b) $Q = 0$.

Equation (22.7) has three real roots of which two are equal.

$$\left.\begin{aligned} (b_{01} + b_{02})_1 &= \frac{B_2}{3} + \frac{3q}{2}, \\ (b_{01} + b_{02})_2 = (b_{01} + b_{02})_3 &= \frac{B_2}{3} - \frac{3q}{2}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (22.10)$$

In this case $\Phi_4(x)$ has equal roots.

1) If the conditions

$$\left.\begin{aligned} B_2 + \frac{3}{8} B_3^2 > 0, \\ p + B_2 B_3^2 + \frac{4}{3} B_2^2 + \frac{3}{16} B_3^4 > 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (22.11)$$

both are valid, all the roots are real and at least two of them are equal.

2) If the conditions (22.11) are not both valid, $\Phi_4(x)$ has two equal real roots and two conjugate complex roots.

Since $\Phi_4(x)$ always has at least two equal roots the interpretation equation generally does not represent a field F, only emanating from sheets. This is, however, still the case, where $\varphi_3(x)$ and $\Phi_4(x)$ have such a common factor that the quotient, obtained by division of $\Phi_4(x)$ with this factor, has only single roots.

c) $Q < 0$.

Equation (22.7) has three real single roots, which can be written

$$\left.\begin{aligned} (b_{01} + b_{02})_1 &= \frac{B_2}{3} + 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\varphi}{3}\right), \\ (b_{01} + b_{02})_2 &= \frac{B_2}{3} + 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right), \\ (b_{01} + b_{02})_3 &= \frac{B_2}{3} + 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right), \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (22.12)$$

where φ is governed by the relationship

$$\cos\varphi = \frac{3q}{2p\sqrt{-\frac{p}{3}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22.13)$$

If $q < 0$, then φ is taken in the first quadrant, and if $q > 0$, then φ is taken in the second quadrant.

As regards the roots of $\Phi_4(x)$ we have to distinguish between two cases, viz.

1) If the conditions (*22.11*) are valid $\Phi_4(x)$ has four unequal real roots. The interpretation equation corresponds generally to four sheets which have $t_0 = 0$.

2) If the conditions (*22.11*) are not both valid $\Phi_4(x)$ has only complex roots. The interpretation equation represents a field F emanating from two sheets with $t_0 \neq 0$.

If the reference line cuts the edge of a sheet ($t_0 = 0$), we have obviously in practice always to reckon with a *thick* sheet by studying a field F within an interval (about three times the breadth of the sheet) situated nearest around the edge. On both sides of this interval it is, however, still possible to represent F with the aid of an interpretation equation of the form (*20.1*). If the calculation points are selected in such a way that they all are situated outside the mentioned interval we may thus apply the interpretation method given above.

In such a case the F-curve corresponding to the interpretation equation may have two or one points of discontinuity where the curve changes from an infinite positive to an infinite negative value. These points of discontinuity are situated at the edge of the sheet.

By working with actual cases of interpretation it is, however, a rare chance that the selected reference line cuts the edge of a sheet. Further, the form of the given F-curve mostly, *a priori*, excludes the mentioned possibility.

On the other hand, it may often occur that the calculations according to the method treated above give results which *inter alia* correspond to sheets with $t_0 = 0$. Such results, however, generally indicate that the calculated F-curve is not in agreement with the observed one. It is true that the two mentioned F-curves coincide in the datum points used for the calculations, but they must at least show considerable differences around the points of discontinuity of the calculated curve.

The facts given above imply that the interpretation calculations generally may give useful results only if the case c) 2 occurs.

The roots of the polynom $\Phi_4(x)$ can be written

$$x = \tfrac{1}{2}\left(b_{11} \pm \sqrt{4b_{01} + b_{11}^2}\right) \text{ and } x = \tfrac{1}{2}\left(b_{12} \pm \sqrt{4b_{02} + b_{12}^2}\right).$$

Hence we may derive the following expressions for the three real roots of $(b_{01} + b_{02})$ given in (*22.12*).

$$\begin{aligned}
(b_{01} + b_{02})_1 &= -\tfrac{1}{2}\left[b_{11}b_{12} - \sqrt{(4b_{01} + b_{11}^2)(4b_{02} + b_{12}^2)}\right],\\
(b_{01} + b_{02})_2 &= b_{01} + b_{02},\\
(b_{01} + b_{02})_3 &= -\tfrac{1}{2}\left[b_{11}b_{12} + \sqrt{(4b_{01} + b_{11}^2)(4b_{02} + b_{12}^2)}\right].
\end{aligned}$$

According to the third equation under (*20.4*), we have

$$-b_{11}b_{12} = B_2 - (b_{01} + b_{02}).$$

On adding the first and third of the expressions above, we thus obtain

$$(b_{01} + b_{02})_1 + (b_{01} + b_{02})_3 = -b_{11}b_{12} = B_2 - (b_{01} + b_{02})_2.$$

It is easy to prove that this relation is in agreement with the formulae (*22.12*).

On summing up the discussion above, we may establish that the interpretation calculations generally may give useful results only if the case c) 2 occurs and that in this case we have to use the value

$$b_{01} + b_{02} = \frac{B_2}{3} + 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right), \quad \ldots\ldots\ldots \quad (22.14)$$

where φ is governed by equation (*22.13*).

§ 23. Thick sheet.

The anomaly field at a thick sheet may in many actual cases be treated as emanating from two thin sheets. It is, however, possible to give a more general and more accurate method of interpretation for this case.

In the following, we treat the common cases where X or Z, or both these components are given along a horizontal profile line, running at right angles to the strike direction of the sheet. Further we assume that the thickness of the sheet has a constant magnitude (B), and that the top side is limited by two horizontal edge-lines, the position of which are determined by the coordinates $x = x_{01}$, $t = t_{01}$ and $x = x_{02}$, $t = t_{02}$ respectively.

If the centre line of the top side has the coordinates $x = 0$ and t and those of another line, situated on the top side and parallel to the centre line, are x_0 and $(t - \Delta t)$ then the components X and Z may be written (comp. 5.1, 5.2)

$$X = -2 \int_{-\frac{B}{2}}^{\frac{B}{2}} \frac{M_{//}(x - x_0) + (t - \Delta t) M_{\perp}}{(t - \Delta t)^2 + (x - x_0)^2} \, d\varepsilon,$$

$$Z = 2 \int_{-\frac{B}{2}}^{\frac{B}{2}} \frac{(t - \Delta t) M_{//} - (x - x_0) M_{\perp}}{(t - \Delta t)^2 + (x - x_0)^2} \, d\varepsilon.$$

We now introduce the symbol φ_e for the angle between the negative x-axis and the inner perpendicular to the top side, and the symbol ξ for the distance between the centre line and the line determined by the coordinates x_0 and $(t - \Delta t)$. If ξ is reckoned positive in the direction towards the edge-line determined by coordinates x_{01}, t_{01} we have

$$x_0 = \xi \sin \varphi_e, \quad \Delta t = -\xi \cos \varphi_e, \quad d\varepsilon = \cos(\varphi_e - \varphi) \cdot d\xi,$$

where φ is the dip angle of the thick sheet.

Further we may state that

$$\frac{\partial X}{\partial \xi} = -\frac{\partial X}{\partial x} \sin \varphi_e + \frac{\partial X}{\partial t} \cos \varphi_e = -\frac{\partial X}{\partial x} \sin \varphi_e - \frac{\partial Z}{\partial x} \cos \varphi_e,$$

$$\frac{\partial Z}{\partial \xi} = -\frac{\partial Z}{\partial x} \sin \varphi_e + \frac{\partial Z}{\partial t} \cos \varphi_e = -\frac{\partial Z}{\partial x} \sin \varphi_e + \frac{\partial X}{\partial x} \cos \varphi_e.$$

Hence

$$-\frac{\partial X}{\partial x} = \frac{\partial X}{\partial \xi} \sin \varphi_e - \frac{\partial Z}{\partial \xi} \cos \varphi_e,$$

$$-\frac{\partial Z}{\partial x} = \frac{\partial X}{\partial \xi} \cos \varphi_e + \frac{\partial Z}{\partial \xi} \sin \varphi_e.$$

If the limit $\frac{B}{2}$ of the above integrals corresponds to the edge-line of coordinates x_{01} and t_{01}, and the limit $-\frac{B}{2}$ corresponds to the edge-line of coordinates x_{02} and t_{02} (i. e. $(x_{01} - x_{02}) \sin \varphi_e \geqslant 0$), we obtain

$$\frac{\partial X}{\partial \xi} = -2 \cos (\varphi_e - \varphi) \left[\frac{(x - x_{01}) M_{//} + t_{01} M_\perp}{t_{01}^2 + (x - x_{01})^2} - \frac{(x - x_{02}) M_{//} + t_{02} M_\perp}{t_{02}^2 + (x - x_{02})^2} \right],$$

$$\frac{\partial Z}{\partial \xi} = 2 \cos (\varphi_e - \varphi) \left[\frac{t_{01} M_{//} - (x - x_{01}) M_\perp}{t_{01}^2 + (x - x_{01})^2} - \frac{t_{02} M_{//} - (x - x_{02}) M_\perp}{t_{02}^2 + (x - x_{02})^2} \right],$$

and consequently

$$\left.\begin{aligned}
-\frac{\partial X}{\partial x} &= -2 \left[\frac{(x - x_{01})(M)_{\rm I} + t_{01}(M)_{\rm II}}{t_{01}^2 + (x - x_{01})^2} - \frac{(x - x_{02})(M)_{\rm I} + t_{02}(M)_{\rm II}}{t_{02}^2 + (x - x_{02})^2} \right],\\
-\frac{\partial Z}{\partial x} &= 2 \left[\frac{t_{01}(M)_{\rm I} - (x - x_{01})(M)_{\rm II}}{t_{01}^2 + (x - x_{01})^2} - \frac{t_{02}(M)_{\rm I} - (x - x_{02})(M)_{\rm II}}{t_{02}^2 + (x - x_{02})^2} \right],\\
\text{where} \quad (M)_{\rm I} &= (M_{//} \sin \varphi_e - M_\perp \cos \varphi_e) \cos (\varphi_e - \varphi)\\
&= B \frac{(x_{01} - x_{02}) M_{//} - (t_{01} - t_{02}) M_\perp}{(t_{01} - t_{02})^2 + (x_{01} - x_{02})^2},\\
(M)_{\rm II} &= (M_{//} \cos \varphi_e + M_\perp \sin \varphi_e) \cos (\varphi_e - \varphi)\\
&= B \frac{(t_{01} - t_{02}) M_{//} + (x_{01} - x_{02}) M_\perp}{(t_{01} - t_{02})^2 + (x_{01} - x_{02})^2},\\
\tan \varphi_e &= \frac{x_{01} - x_{02}}{t_{01} - t_{02}}.
\end{aligned}\right\} \quad \ldots \quad (23.1)$$

The above expressions of $(M)_{\rm I}$ and $(M)_{\rm II}$ give

$$\left.\begin{aligned}
BM_{//} &= (x_{01} - x_{02}) (M)_{\rm I} + (t_{01} - t_{02}) (M)_{\rm II},\\
BM_\perp &= -(t_{01} - t_{02}) (M)_{\rm I} + (x_{01} - x_{02}) (M)_{\rm II}.
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (23.2)$$

The formulae (23.1) of $-\frac{\partial X}{\partial x}$ and $-\frac{\partial Z}{\partial x}$ have the same form as those of the components X and Z respectively, which are valid for the anomaly from two thin sheets with the edges coinciding with the edge lines of the top side of the thick sheet. The only difference is that the quantities $(M)_{\mathrm{I}}$ and $(M)_{\mathrm{II}}$ are substituted for $\varepsilon M_{/\!/}$ respectively $\varepsilon M_{\perp}$.

By using the derivatives of X or Z as regards x instead of the components themselves the interpretation of the anomaly at a thick sheet is thus transferred into the earlier treated problem of interpreting the anomaly of X and Z at two thin sheets. It is, however, so far a special case as the interpretation problem includes the stipulation that the values of $\varepsilon M_{/\!/}$ and $\varepsilon M_{\perp}$ which are valid for one of the thin sheets are numerically the same but have opposite signs from those of the other sheet.

In the following, we treat a convenient method of calculation for the case where both $\frac{\partial X}{\partial x}$ and $\frac{\partial Z}{\partial x}$ are known.

Given $\frac{\partial X}{\partial x}$ and $\frac{\partial Z}{\partial x}$.

On putting $-\frac{\partial X}{\partial x} = X'$, $-\frac{\partial Z}{\partial x} = Z'$, and solving $(M)_{\mathrm{I}}$ and $(M)_{\mathrm{II}}$ from the first two equations (23.1), we obtain

$$\left.\begin{aligned} 2(M)_{\mathrm{I}} &= \frac{Z'(C_0x^2 + C_1x + C_2) - X'(D_0x^2 + D_1x + D_2)}{C_0^2 + D_0^2}, \\ 2(M)_{\mathrm{II}} &= \frac{X'(C_0x^2 + C_1x + C_2) + Z'(D_0x^2 + D_1x + D_2)}{C_0^2 + D_0^2}, \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23.3)$$

where

$$\left.\begin{aligned} C_0 &= t_{01} - t_{02}, & D_0 &= x_{01} - x_{02}, \\ C_1 &= 2(x_{01}t_{02} - x_{02}t_{01}), & D_1 &= -(t_{01}^2 + x_{01}^2) + (t_{02}^2 + x_{02}^2), \\ C_2 &= t_{01}(t_{02}^2 + x_{02}^2) - t_{02}(t_{01}^2 + x_{01}^2), & D_2 &= -x_{01}(t_{02}^2 + x_{02}^2) + x_{02}(t_{01}^2 + x_{01}^2). \end{aligned}\right\} (23.4)$$

Further the following relations are valid:

$$\left.\begin{aligned} C_1 &= -(x_{01} + x_{02})C_0 + (t_{01} + t_{02})D_0, \\ D_1 &= -(t_{01} + t_{02})C_0 - (x_{01} + x_{02})D_0, \\ C_2 &= (x_{01}x_{02} - t_{01}t_{02})C_0 - (t_{01}x_{02} + t_{02}x_{01})D_0, \\ D_2 &= (t_{01}x_{02} + t_{02}x_{01})C_0 + (x_{01}x_{02} - t_{01}t_{02})D_0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (23.5)$$

We now insert the expressions (23.3) of $(M)_{\mathrm{I}}$ and $(M)_{\mathrm{II}}$ in the formulae (23.2). On putting C_0 and D_0 instead of $(t_{01} - t_{02})$ respectively $(x_{01} - x_{02})$ in the latter equations and using the expressions (23.5) of C_1, C_2, D_1, D_2 in the former equations, we obtain

$$\left.\begin{aligned} 2BM_{/\!/} &= -X'\left[x^2 - x(x_{01} + x_{02}) + (x_{01}x_{02} - t_{01}t_{02})\right] \\ &\quad + Z'\left[x(t_{01} + t_{02}) - (t_{01}x_{02} + t_{02}x_{01})\right], \\ 2BM_{\perp} &= -Z'\left[x^2 - x(x_{01} + x_{02}) + (x_{01}x_{02} - t_{01}t_{02})\right] \\ &\quad - X'\left[x(t_{01} + t_{02}) - (t_{01}x_{02} + t_{02}x_{01})\right]. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23.6)$$

The two equations above contain six unknown quantities which may be solved by using three datum points (1, 2, 3).

On putting

$$\left.\begin{aligned}
g_i &= \frac{x_1^2[X_1'(X_i'-X_1')+Z_1'(Z_i'-Z_1')]-x_i^2[X_i'(X_i'-X_1')+Z_i'(Z_i'-Z_1')]}{(X_i'-X_1')^2+(Z_i'-Z_1')^2},\\
h_i &= \frac{x_1[X_1'(X_i'-X_1')+Z_1'(Z_i'-Z_1')]-x_i[X_i'(X_i'-X_1')+Z_i'(Z_i'-Z_1')]}{(X_i'-X_1')^2+(Z_i'-Z_1')^2},\\
k_i &= l_i(x_i+x_1), \quad l_i = \frac{(x_i-x_1)(X_i'Z_1'-X_1'Z_i')}{(X_i'-X_1')^2+(Z_i'-Z_1')^2}.
\end{aligned}\right\} \quad (23.7)$$

we may calculate four of the unknown quantities in (23.6) with the aid of the following formulae.

$$\left.\begin{aligned}
x_{01}+x_{02} &= \frac{(g_2-g_3)(h_2-h_3)+(k_2-k_3)(l_2-l_3)}{(h_2-h_3)^2+(l_2-l_3)^2},\\
t_{01}+t_{02} &= \frac{(h_2-h_3)(k_2-k_3)-(g_2-g_3)(l_2-l_3)}{(h_2-h_3)^2+(l_2-l_3)^2},\\
x_{01}x_{02}-t_{01}t_{02} &= g_i-h_i(x_{01}+x_{02})+l_i(t_{01}+t_{02}),\\
t_{01}x_{02}+t_{02}x_{01} &= k_i-l_i(x_{01}+x_{02})-h_i(t_{01}+t_{02}).
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23.8)$$

The quantities $BM_{//}$ and $BM_{\perp}$ are then obtained from (23.6).

For determination of C_0 and D_0 we note that

$$\begin{aligned}
4(x_{01}x_{02}-t_{01}t_{02}) &= [(x_{01}+x_{02})+D_0][(x_{01}+x_{02})-D_0]\\
&\quad +[(t_{01}+t_{02})+C_0][(t_{01}+t_{02})-C_0]\\
&= (x_{01}+x_{02})^2-(t_{01}+t_{02})^2+C_0^2-D_0^2,\\
4(t_{01}x_{02}+t_{02}x_{01}) &= [(t_{01}+t_{02})+C_0][(x_{01}+x_{02})-D_0]\\
&\quad +[(x_{01}+x_{02})+D_0][(t_{01}+t_{02})-C_0]\\
&= 2(x_{01}+x_{02})(t_{01}+t_{02})-2C_0D_0,
\end{aligned}$$

and put

$$\left.\begin{aligned}
p &= C_0^2-D_0^2 = 4(x_{01}x_{02}-t_{01}t_{02})+(t_{01}+t_{02})^2-(x_{01}+x_{02})^2,\\
q &= 2C_0D_0 = -4(t_{01}x_{02}+t_{02}x_{01})+2(x_{01}+x_{02})(t_{01}+t_{02}).
\end{aligned}\right\} \quad \ldots\ldots \quad (23.9)$$

On squaring and adding these two expressions we obtain

$$(C_0^2+D_0^2)^2 = p^2+q^2 \quad \text{and thus} \quad C_0^2+D_0^2 = \sqrt{p^2+q^2}.$$

The above expression and the relations (23.9) give

$$D_0 = \pm\sqrt{\frac{-p+\sqrt{p^2+q^2}}{2}}, \quad C_0 = \frac{q}{2D_0}, \quad \tan\varphi_e = \frac{-p+\sqrt{p^2+q^2}}{q}. \quad (23.10)$$

Further we obviously have

$$\left.\begin{aligned}
x_{01} &= \tfrac{1}{2}[(x_{01}+x_{02})+D_0], \quad & t_{01} &= \tfrac{1}{2}[(t_{01}+t_{02})+C_0],\\
x_{02} &= \tfrac{1}{2}[(x_{01}+x_{02})-D_0], \quad & t_{02} &= \tfrac{1}{2}[(t_{01}+t_{02})-C_0].
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (23.11)$$

By deriving the formulae (*23.1*) we have assumed that $(x_{01} - x_{02}) \sin \varphi_e \geq 0$. Consequently D_0 shall be positive if $0 \leq \varphi_e \leq \pi$, and negative if $\pi \leq \varphi_e \leq 2\pi$. Whether we choose the positive or the negative value of D_0 in (*23.10*), we obtain, however, the same position of the top side of the thick sheet, since the only difference in the results calculated from (*23.11*) is that x_{01} and x_{02} as well t_{01} and t_{02} exchange values.

The expression (*23.10*) of $\tan \varphi_e$ gives two values of φ_e which can be written φ'_e and $\varphi''_e = \varphi'_e + \pi$, where $0 \leq \varphi'_e \leq \pi$. These two values of φ_e satisfy both the primary equations (*23.1*). This is apparent from the fact mentioned above regarding x_{01}, x_{02}, t_{01}, t_{02} and the relations (*23.2*) which imply that $(M)_{\text{I}}$ and $(M)_{\text{II}}$ change signs if φ'_e is changed in φ''_e.

The values of $BM_{//}$ and $BM_{\perp}$, obtained from the formulae (*23.6*), may be used for further calculations concerning the quantities $\varkappa$, φ and the natural remanence of the thick sheet. Thereby we may use the formulae given in § 8, if we put $\varepsilon = B$, where

$$B = \left| (x_{01} - x_{02}) \sin \varphi \right|.$$

Relation figure.

On solving x and x^2 from the two equations (*23.6*), we may obtain two different expressions for x^2, which give the equation

$$\left.\begin{aligned} &4(a_{X'} BM_{\perp} - a_{Z'} BM_{//})^2 + 2(x_{01} - x_{02})(t_{01} - t_{02})(a_{X'} BM_{\perp} - a_{Z'} BM_{//}) \\ &+ 2(t_{01} + t_{02})(a_{X'} BM_{//} + a_{Z'} BM_{\perp}) = t_{01} t_{02} \left[(t_{01} + t_{02})^2 + (x_{01} - x_{02})^2 \right], \\ &\text{where} \quad a_{X'} = \frac{X'}{X'^2 + Z'^2}, \quad a_{Z'} = \frac{Z'}{X'^2 + Z'^2}. \end{aligned}\right\} \quad \ldots (23.12)$$

Consequently, the relation between $a_{X'}$ and $a_{Z'}$ may be illustrated by a curve of the second degree in the same way as earlier described (comp. p. 24) regarding X and Z in the case of a thin sheet.

The above equation may be written

$$\left[(a_{X'} BM_{\perp} - a_{Z'} BM_{//}) + \frac{(x_{01} - x_{02})(t_{01} - t_{02})}{4} \right]^2$$

$$+ \frac{(t_{01} + t_{02})^2}{2} \left[(a_{X'} BM_{//} + a_{Z'} BM_{\perp}) - \frac{(x_{01} - x_{02})^2 + 4 t_{01} t_{02}}{8} \right] = 0. \quad \ldots\ldots (23.13)$$

Hence it is clear that the *relation figure* for $a_{X'}$ and $a_{Z'}$ is a parabola. The axis of the parabola forms an angle φ_p with the $a_{X'}$— axis and the vertex of the parabola has the coordinates $a_{X'_0}$ and $a_{Z'_0}$. These quantities may be calculated from the formulae

$$\left.\begin{aligned} &\tan \varphi_p = \frac{M_{\perp}}{M_{//}}, \\ &a_{X'_0} = \frac{BM_{//}\left[(x_{01} - x_{02})^2 + 4 t_{01} t_{02}\right] - 2 BM_{\perp} (x_{01} - x_{02})(t_{01} - t_{02})}{8 (BM_T)^2}, \\ &a_{Z'_0} = \frac{2 BM_{//} (x_{01} - x_{02})(t_{01} - t_{02}) + BM_{\perp}\left[(x_{01} - x_{02})^2 + 4 t_{01} t_{02}\right]}{8 (BM_T)^2}. \end{aligned}\right\} \quad \ldots (23.14)$$

CHAPTER VI.

Actual Examples.

In order to illustrate the actual application of the methods of interpretation deduced in the foregoing, some examples of the interpretation of anomalies from a number of iron ore deposits will be given.

First we treat three Z-profiles given in a paper by M. Rössiger and K. Puzischa: "Magnetic Measurements on Diabase Formations in the Oberhartz Mountains" (19). The three chosen profiles are designated Spitzenberg I and II and Eisener Weg II. In all three cases the anomalies originate from thin sheet-like ore-bodies consisting of magnetite derived from hematite.

Further an interpretation is made of a profile curve of the total intensity measured by an airborne magnetometer across the Benson Mines, Adirondac district, U. S. A. Finally we treat two published profile curves representing the vertical and horizontal intensities along a profile line across the vast Russian iron ore deposit in the Kursk district.

In the interpretation work it should not be necessary to calculate the magnetic parameters of the sheets with greater accuracy than what can be useful in practice. It is often difficult, however, to judge a priori what accuracy this condition corresponds to as regards the parameters of the interpretation equation. Further the checking possibilities in the calculations are much better if the values of the last mentioned parameters are so exactly computed that they almost exactly satisfy the interpretation equation in all the calculation points. For that reason the calculations are made with great exactness. Essential approximations are only introduced into the final results.

§ 24. Spitzenberg II.

In Fig. 24.1 an interpretation diagram is shown where the primary data are taken from figs 27 and 31 (p. 89 and p. 94) in the above-mentioned paper (19). The magnetic profile is measured at right angles to the strike of the sheet and the edge of the sheet can be considered horizontal ($\alpha = \gamma = 0$). The measured values of Z are marked out by open circles. The terrain profile is a straight line having a slope angle $\beta = 6°$.

The points of observation in the interpretation diagram indicate that the course of the magnetic profile curve is characteristic of the anomaly at a single sheet (comp. § 9). Consequently it is possible to use the interpretation equation

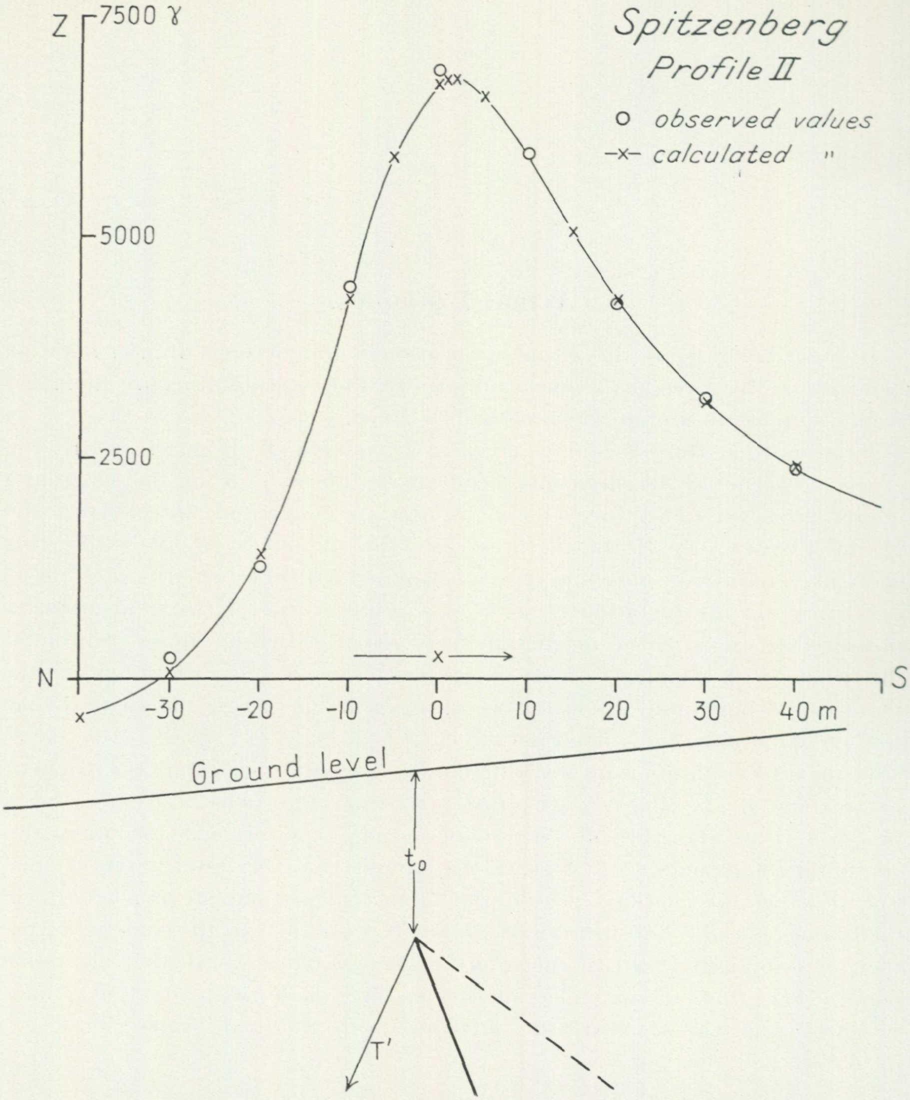

Fig. 24.1. Interpretation diagram for a *Z*-profile across a sheet-like iron ore-body at Spitzenberg in Harz, Germany. The open circles give the *Z*-values observed at the ground. The curve line corresponds to the calculated values of the magnetic parameters of the ore-sheet. The real position of the sheet is marked by the heavy line. The dashed line shows the calculated position of the sheet assuming only induced magnetization caused by T'.

(*6.1*) by the calculations if we assume that given data of Z emanate from a sheet of infinite length.

In order to neutralize as much as possible the influence of punctual errors on the calculated result one had better include all observed data in the calculations. Hence we place the eight given observation points in two groups to the arrangement below

I		II	
$x_1 = -20$ m	$Z_1 = 1\,260$ gammas	$x_1 = -30$ m	$Z_1 = 240$ gammas
$x_2 = 0$ »	$Z_2 = 6\,880$ »	$x_2 = -10$ »	$Z_2 = 4\,440$ »
$x_3 = 20$ »	$Z_3 = 4\,240$ »	$x_3 = 10$ »	$Z_3 = 5\,940$ »
$x_4 = 40$ »	$Z_4 = 2\,360$ »	$x_4 = 30$ »	$Z_4 = 3\,200$ »

and calculate x_0, t_0, $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ from both groups I and II.

For the abscissae of both groups

$$x_4 - x_3 = x_3 - x_2 = x_2 - x_1 = 20 \text{ m}.$$

We may thus carry out the calculations according to case 3) in § 16. The parameters of the interpretation equation may then be computed in the following way:

I

$$n_2 = \frac{Z_2}{Z_1 - Z_2} = -1.2242,$$

$$n_3 = \frac{Z_3}{Z_1 - Z_3} = -1.4228,$$

$$n_4 = \frac{Z_4}{Z_1 - Z_4} = -2.1455,$$

$$x_1 = -x_3, \quad x_2 = 0, \quad x_4 = 2x_3,$$

$$x_1' = \frac{-4n_3 + 6n_4}{n_2 - 4n_3 + 3n_4} x_3 = 3.6465\, x_3,$$

$$a_0 = -x_3 x_1' n_2 Z_1 = 4.464\, x_3^2 Z_1,$$

$$a_1 = [n_2 x_1' + 2n_3(x_3 - x_1')]Z_1 = 3.067\, x_3 Z_1,$$

$$b_0 = -\frac{a_0}{Z_1} + x_1' x_3 = -0.818\, x_3^2,$$

$$b_1 = -\frac{a_1}{Z_1} - x_3 + x_1' = -0.420\, x_3,$$

II

$$n_2 = \frac{Z_2}{Z_1 - Z_2} = -1.0571,$$

$$n_3 = \frac{Z_3}{Z_1 - Z_3} = -1.0421,$$

$$n_4 = \frac{Z_4}{Z_1 - Z_4} = -1.0811,$$

$$x_1 = -3x_3, \quad x_2 = -x_3, \quad x_4 = 3x_3,$$

$$x_1' = \frac{-n_2 - 4n_3 + 9n_4}{n_2 - 4n_3 + 3n_4} x_3 = 34.124\, x_3,$$

$$a_0 = x_3^2\left[-\left(1 + \frac{x_1'}{x_3}\right)n_2 + 2\left(1 - \frac{x_1'}{x_3}\right)n_3\right]Z_1 = 106.17\, x_3^2 Z_1,$$

$$a_1 = x_3\left[n_2\left(1 + \frac{x_1'}{x_3}\right) + 2n_3\left(1 - \frac{x_1'}{x_3}\right)\right]Z_1 = 31.907\, x_3 Z_1,$$

$$b_0 = -\frac{a_0}{Z_1} + 3x_1' x_3 = -3.795\, x_3^2,$$

$$b_1 = -\frac{a_1}{Z_1} - 3x_3 + x_1' = -0.783\, x_3.$$

The magnetic parameters of the ore-body may now be calculated with the aid of formulae (7.7). Since $\alpha = \gamma = 0$ and $v_x = v_y = 0$, $v_z = 1$ we have $v_x' = 0$, $v_z' = 1$, $p = \tan\beta$ and $\mu^2 = 1 + \tan^2\beta$. We may therefore write

$$\left.\begin{aligned} x_0 &= \tfrac{1}{2}\left[b_1 + \tan\beta\sqrt{-4b_0 - b_1^2}\right], \quad \varepsilon M_{/\!/} = \tfrac{1}{2}\left[a_1 \tan\beta + \frac{2a_0 + a_1 b_1}{\sqrt{-4b_0 - b_1^2}}\right], \\ t_0 &= \frac{1+\tan^2\beta}{2}\sqrt{-4b_0 - b_1^2}, \qquad \varepsilon M_{\perp} = \tfrac{1}{2}\left[-a_1 + \frac{2a_0 + a_1 b_1}{\sqrt{-4b_0 - b_1^2}}\tan\beta\right]. \end{aligned}\right\} \quad (24.1)$$

As mentioned earlier we have $\beta = 6°$ for groups I and II, and thus $\tan\beta = 0.105$ and $\tan^2\beta = 0.011$.

Subsequent calculations are carried out according to the following plan:

I

$$\sqrt{-4b_0 - b_1^2} = 1.759\, x_3,$$
$$2a_0 + a_1 b_1 = 7.639\, x_3^2 Z_1,$$
$$\frac{2a_0 + a_1 b_1}{\sqrt{-4b_0 - b_1^2}} = 4.343\, x_3 Z_1,$$
$$x_3 = 20 \text{ m},$$
$$Z_1 = 1260 \cdot 10^{-5} \text{ Gauss},$$
$$x_0 = -\frac{0.237}{2} x_3 = -2.37 \text{ m},$$
$$t_0 = \frac{1.78}{2} x_3 = 17.8 \text{ m},$$
$$\varepsilon M_{/\!/} = \frac{4.664}{2} x_3 Z_1 = 0.588 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = -\frac{2.613}{2} x_3 Z_1 = -0.329 \text{ m} \cdot \text{Gauss},$$

II

$$\sqrt{-4b_0 - b_1^2} = 3.817\, x_3,$$
$$2a_0 + a_1 b_1 = 187.4\, x_3^2 Z_1,$$
$$\frac{2a_0 + a_1 b_1}{\sqrt{-4b_0 - b_1^2}} = 49.09\, x_3 Z_1,$$
$$x_3 = 10 \text{ m},$$
$$Z_1 = 240 \cdot 10^{-5} \text{ Gauss};$$
$$x_0 = -\frac{0.384}{2} x_3 = -1.92 \text{ m},$$
$$t_0 = \frac{3.86}{2} x_3 = 19.3 \text{ m},$$
$$\varepsilon M_{/\!/} = \frac{52.42}{2} x_3 Z_1 = 0.629 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = -\frac{26.78}{2} x_3 Z_1 = -0.321 \text{ m} \cdot \text{Gauss}.$$

For I and II we obtain the mean values

$$\left.\begin{aligned} x_0 &= -2.15 \pm 0.23 \text{ m}, \quad \varepsilon M_{/\!/} = 0.608 \pm 0.021 \text{ m} \cdot \text{Gauss}, \\ t_0 &= 18.5 \pm 0.8 \text{ m}, \quad \varepsilon M_{\perp} = -0.325 \pm 0.004 \text{ m} \cdot \text{Gauss}. \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (24.2)$$

In order to visualize the Z-curve corresponding to these mean values we calculate a number of points on it. For this we use the formula

$$Z = \frac{a_0 + a_1 x}{-b_0 - b_1 x + x^2}.$$

According to (24.1) the parameters in this formula are determined by the following mean-values of the quantities given above for the groups I and II.

$$a_1 = 0.7693 \text{ m} \cdot \text{Gauss}, \qquad \sqrt{-4b_0 - b_1^2} = 36.68 \text{ m},$$

$$b_1 = -8.12 \text{ m} \qquad \frac{2a_0 + a_1 b_1}{\sqrt{-4b_0 - b_1^2}} = 1.136 \text{ m} \cdot \text{Gauss}.$$

Hence we obtain

$$b_0 = -\left(\frac{36.68}{2}\right)^2 - \left(\frac{8.12}{2}\right)^2 = -352.8 \text{ m}^2,$$

$$a_0 = \frac{0.7693 \cdot 8.12 + 36.68 \cdot 1.136}{2} = 23.96 \text{ m}^2 \cdot \text{Gauss}.$$

and

$$Z = \frac{23.96 + 0.7693\,x}{352.8 + 8.12\,x + x^2}\ 10^5 \text{ gammas}.$$

The curve drawn in Fig. 24.1 has been calculated with the aid of the formula above. The coordinates of the points designated by (×) are given in the table below. There are also given the differences between observed (Z_{obs}) and calculated (Z_{calc}) values of Z.

x	Z_{calc}	Z_{obs}	$Z_{obs}-Z_{calc}$	x	Z_{calc}	Z_{obs}	$Z_{obs}-Z_{calc}$
—40 m	—418			1 m	6 833		
—30	87	240	153	2	6 835		
—20	1 452	1 260	—192	5	6 646		
—15	2 724			10	5 928	5 940	12
—10	4 378	4 440	62	15	5 074		
—5	5 965			20	4 299	4 240	—59
0	6 791	6 880	89	30	3 144	3 200	56
				40	2 403	2 360	—43

The extreme points on the Z-curve may be calculated from the formulae (9.1). We obtain

$$x_e = -\frac{a_0}{a_1} \pm \sqrt{\left(\frac{a_0}{a_1}\right)^2 + b_1\left(\frac{a_0}{a_1}\right) - b_0} = -31.15 \pm 32.71 \text{ m}.$$

Hence the coordinates of the maximum point (M) and the minimum point (m).

$$x_M = 1.56 \text{ m}$$

$$Z_M = \frac{a_1}{2x_M - b_1} = \frac{0.7693}{11.24} \cdot 10^5 = 6\,844 \text{ gammas}$$

$$x_m = -63.86 \text{ m}$$

$$Z_m = \frac{a_1}{2x_m - b_1} = \frac{0.7693}{-119.60} \cdot 10^5 = -643 \text{ gammas}.$$

The calculated curve conforms rather well with the observed points. The differences between Z_{obs} and Z_{calc} for the six southernmost points of measurements are less then 2 % of the observed values of Z. The corresponding differences for the two nothernmost points of observation are, however, rather great and of opposite signs. It seems probable that this defective agreement is caused by a strange disturbing field.

In the preceding we have seen that the values (*24.2*) of the magnetic parameters satisfy rather well the observed profile of Z. It remains to investigate whether these values are also in agreement with the data of the ore-body.

According to the earlier mentioned paper (19) the values

$$x_0 \approx 0, \quad t_0 \approx 18 \text{ m}, \quad \varepsilon \approx 3.5 \text{ m}$$

are valid for the ore sheet. The susceptibility ($\varkappa$) is determined for two samples. For these the values 0.44 and 0.49 are given. In the following we use the value

$$\varkappa = 0.46.$$

The ore sheet dips 68° to the south *i. e.*

$$\varphi = 180° - 68° = 112°.$$

For the earth's normal magnetic field we have

$$T = 0.478 \text{ Gauss}, \quad i = 66°.5, \quad \vartheta = 165°.$$

As $\alpha = \gamma = 0$ we obtain from the formulae (*3.2*)

$$T' = T\sqrt{1 - \cos^2 i \sin^2 \vartheta} = 0.475 \text{ Gauss},$$

$$\tan i' = -\frac{\tan i}{\cos \vartheta} = 2.38, \quad i.\,e.\ i' = 67°.2.$$

According to the values (*24.2*) the edge of the ore-sheet is situated approximately 2 m to the north of the above given position. The calculated depth of the mentioned edge practically coincides with the given value. In the paper it is not stated, however, with what degree of reliability the position of the ore-sheet edge has been determined.

Calculations proceding from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$.

The length of the ore-body seems to be at least 100 m. The depth to the lower edge of the ore-sheet is unknown. Applying the formulae (*3.3*) for $N_{//}$ and $N_{\perp}$ and putting $2l = 100$ m we obtain for

1) $d = \infty \quad N_{//} = 0, \quad N_{\perp} = 0.966 \cdot 4\pi,$
2) $d = 100$ m, $\quad N_{//} = 0.033 \cdot 4\pi, \quad N_{\perp} = 0.935 \cdot 4\pi.$

Further we obtain with the aid of the formulae (*8.4*) in case

1) $\tan(i'' - \varphi') = -3.52, \quad i'' - \varphi' = i'' - \varphi = -74°.1,$
$\varepsilon T'' = 4.85 \text{ m} \cdot \text{Gauss},$
2) $\tan(i'' - \varphi') = -2.87, \quad i'' - \varphi' = i'' - \varphi = -70°.8,$
$\varepsilon T'' = 4.80 \text{ m} \cdot \text{Gauss}.$

As the calculated values in the cases 1) and 2), which represent extreme cases, show only small differences we use in the following the mean values

$$i'' - \varphi = -72° \quad \text{and} \quad \varepsilon T'' = 4.82 \text{ m} \cdot \text{Gauss}.$$

Starting from the known $\varkappa$-value and the assumption that no remanent magnetization exists in the ore-body ($i'' = i'$, $T'' = T'$) we thus have

$$\varphi = i' + 72° = 140° \quad \text{and} \quad \varepsilon = 4.82 : T' = 10.2 \text{ m}.$$

These results imply that the ore-sheet should have a dip of 40° to the south (see Fig. 24.1) as against a given value of 68° to the south and a thickness of 10.2 m as against a given value of only 3.5 m. It is therefore highly probable that the ore-sheet in fact possesses a remanent magnetization.

It is possible to calculate this remanent magnetization by using also the given values of ε and φ. Thereby we compute at first from the above expressions

$$i'' = \varphi - 72° = 40° \quad \text{and} \quad T'' = 4.82 : \varepsilon = 1.38 \text{ Gauss.}$$

Thus $T'' : T' = 2.90$, $\cos\,(i'' - i') = \cos\,(-27°) = 0.891$.

$$\sin\,i'' = 0.643, \quad \sin\,i' = 0.921$$
$$\cos\,i'' = 0.766, \quad \cos\,i' = 0.391.$$

Inserting these values into the formulae (*8.3*) we obtain

$$k' = 2.06, \quad \tan\,\psi' = 0.515, \quad \psi' = 27°.$$

Hence $I_r{}' = k'\varkappa T' = 0.45$ Gauss, $\varphi - \psi' = 85°$.

Thus the natural remanence has an intensity of 0.45 Gauss and its direction is approximately perpendicular to the plane of the ore-sheet.

Finally some remarks as regards the demagnetizing factors and the remanent magnetization:

In the formulae (*8.1*) we have introduced the effect of the remanent magnetization by using the quantities T'' and i'' of the apparent magnetizing field instead of the quantities T' and i' of the real magnetizing field. In an analogous way we may from the primary equations (*3.1*) derive the formulae

$$\left.\begin{aligned} \varepsilon\varkappa T' \cos\,(i' - \varphi') &= \varepsilon M_{//}(1 + \varkappa N'_{//}), \\ \varepsilon\varkappa T' \sin\,(i' - \varphi') &= \varepsilon M_{\perp}(1 + \varkappa N'_{\perp}), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24.3)$$

where $N'_{//}$ and $N'_{\perp}$ are *apparent demagnetizing factors.*

In the actual case the remanent magnetization is perpendicular to the plane of the ore-sheet. Consequently, only the magnitude of $\varepsilon M_{\perp}$ is influenced by this magnetization and thus $N_{//}' = N_{//}$. On assuming $\varkappa N_{//} \ll 1$ and considering φ' as known, we obtain from the relations (*24.3*)

$$\varkappa N_{\perp}' = \frac{\varepsilon M_{//}}{\varepsilon M_{\perp}} \tan\,(i' - \varphi') - 1 = 0.673,$$

$$\varepsilon\varkappa = \frac{\varepsilon M_{//}}{T' \cos\,(i' - \varphi')} = 1.72 \text{ m.}$$

Fixing the thickness of the sheet to 3.5 m these relations give

$$\varkappa = 0.49, \quad N'_{\perp} = 1.37 = 0.44\pi.$$

Hence it is evident that the remanent magnetization of the ore-sheet has the same effect as if the demagnetizing factor $N_{\perp}$ were diminished to about one ninth of its real value.

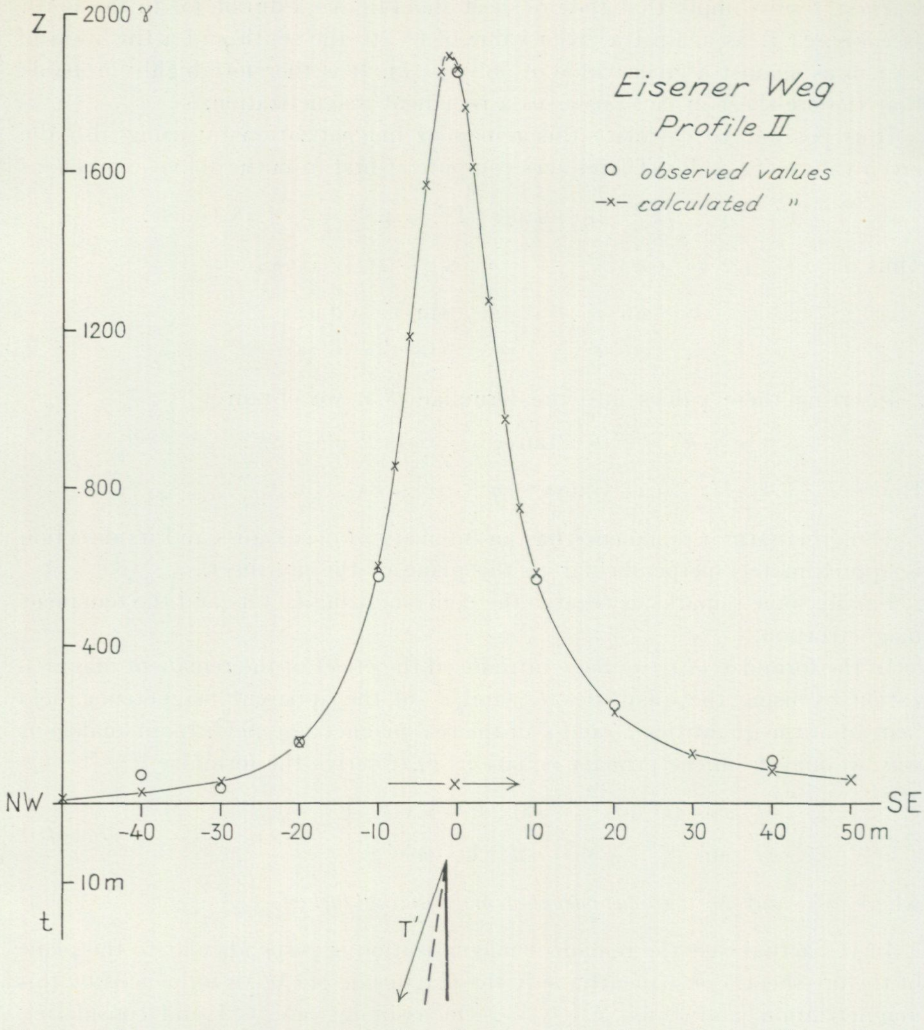

Fig. 25.1. Interpretation diagram for a Z-profile across a sheet-like ore-body at Eisener Weg in Harz, Germany. The diagram shows the calculation results according to case I. The curve line corresponds to the calculated values of the magnetic parameters of the ore-sheet. The real position of the sheet is marked by the heavy line. The dashed line shows the calculated position of the sheet assuming only induced magnetization caused by T'.

§ 25. Eisener Weg II.

Fig. 25.1 shows an interpretation diagram that has been drawn from Fig. 19 on page 200 of Rössiger and Puzicha's paper (19). The terrain profile coincides in the diagram with the x-axis and $\alpha = \beta = \gamma = 0$. The measured values of Z are indicated by open circles. The authors state that there is probably a regional anomaly in the vertical intensity of approximately 100 gammas in the locality of the ore-body. The points in the diagram are not, however, reduced by any such anomaly.

In the following, there are first given the results of an interpretation calculation based on the given observed data, without any correction for the regional anomaly having been added to the Z-values. The use of the method of least squares is thereby demonstrated for calculating the parameters of the interpretation equation (*6.1*). Subsequently we apply the equation (*12.3*) for a simultaneous calculation of ΔZ and the magnetic parameters of the ore-sheet.

I. *Calculation of the parameters of the interpretation equation* (*6.1*) *according to the method of least squares.*

If we postulate that all datum points in the magnetic profile are of equal importance, we have to determine a_0, a_1, b_0, b_1 so that

$$\Sigma(Z_{\text{calc}} - Z_{\text{obs}})^2$$

is a minimum. The summation should include in this instance, as in the following, all given datum points.

In order to complete the calculations we first choose suitable proximate values a_0', a_1', b_0', b_1' with the aid of calculations for four of the datum points. We can then write

$$a_0 = a_0' + \Delta a_0, \qquad b_0 = b'_0 + \Delta b_0,$$
$$a_1 = a_1' + \Delta a_1, \qquad b_1 = b'_1 + \Delta b_1.$$

It is thus required to calculate the four increments Δa_0, Δa_1, Δb_0, Δb_1.

The following notations are introduced

$$Z_{i_{\text{calc}}} = \frac{a_0 + a_1 x_i}{-b_0 - b_1 x_i + x_i^2}, \qquad Z_i' = \frac{a_0' + a_1' x_i}{-b_0' - b_1' x_i + x_i^2},$$

$$l_i = Z_{i_{\text{obs}}} - Z_i', \qquad v_i = Z_{i_{\text{calc}}} - Z_{i_{\text{obs}}},$$

$$c_i = \frac{\partial Z_i'}{\partial a_0'} = \frac{1}{-b_0' - b_1' x_i + x_i^2}, \quad d_i = \frac{\partial Z_i'}{\partial a_1'} = c_i x_i,$$

$$e_i = \frac{\partial Z_i'}{\partial b_0'} = c_i Z_i', \qquad f_i = \frac{\partial Z_i'}{\partial b_i'} = c_i x_i Z_i'.$$

When the increments $\Delta a_0 \,..\, \Delta b_1$ are small the error equation holds as follows:

$$Z_{i_{\text{calc}}} - Z_i' = l_i + v_i = c_i \Delta a_0 + d_i \Delta a_1 + e_i \Delta b_0 + f_i \Delta b_1.$$

By making use of the summation notation [ ], introduced by Gauss in his theory of probability, the condition that

$$\Omega = [vv] = \Sigma(Z_{i_{\text{calc}}} - Z_{i_{\text{obs}}})^2 = \Sigma(c_i \Delta a_0 + d_i \Delta a_1 + e_i \Delta b_0 + f_i \Delta b_1 - l_i)^2$$

is to be a minimum may be expressed as follows:

$$\tfrac{1}{2}\frac{\partial \Omega}{\partial(\Delta a_0)} = [cv] = 0, \qquad \tfrac{1}{2}\frac{\partial \Omega}{\partial(\Delta b_0)} = [ev] = 0,$$

$$\tfrac{1}{2}\frac{\partial \Omega}{\partial(\Delta a_1)} = [dv] = 0, \qquad \tfrac{1}{2}\frac{\partial \Omega}{\partial(\Delta b_1)} = [fv] = 0.$$

Further it is obvious that

$$\tfrac{1}{2}\frac{\partial Z_{\text{calc}}}{\partial \Omega} = \Sigma(Z_{i_{\text{calc}}} - Z_{i_{\text{obs}}}) = [v] = 0.$$

On multiplying both sides of the error equation by c_i, d_i, e_i, f_i, l_i respectively and adding each of these equations over all given datum points we obtain

$$[cc]\,\Delta a_0 + [cd]\,\Delta a_1 + [ce]\,\Delta b_0 + [cf]\,\Delta b_1 = [cl],$$
$$[dc]\,\Delta a_0 + [dd]\,\Delta a_1 + [de]\,\Delta b_0 + [df]\,\Delta b_1 = [dl],$$
$$[ec]\,\Delta a_0 + [ed]\,\Delta a_1 + [ee]\,\Delta b_0 + [ef]\,\Delta b_1 = [el],$$
$$[fc]\,\Delta a_0 + [fd]\,\Delta a_1 + [fe]\,\Delta b_0 + [ff]\,\Delta b_1 = [fl],$$
$$[lc]\,\Delta a_0 + [ld]\,\Delta a_1 + [le]\,\Delta b_0 + [lf]\,\Delta b_1 + [vv] = [ll].$$

The five equations above are the *normal equations* necessary for the calculations. From the first four equations we obtain Δa_0, Δa_1, Δb_0, Δb_1 and then $[vv]$ from the fifth equation. For the actual evaluation it is convenient to use the well-known method developed by Gauss.

The Z-profile under consideration has been treated according to the method detailed above. Thereby the following proximate values have been used

$$a_0' = 0.8 \text{ m}^2\cdot\text{Gauss}, \qquad b_0' = -50 \text{ m}^2,$$
$$a_1' = 0.02 \text{ m}\cdot\text{Gauss}, \qquad b_1' = 0.$$

Since the evaluation comprises a fairly large volume of calculation only the final results are noted below, as follows:

$$a_0 = a_0' + \Delta a_0 = 0.8 + 0.08286 = 0.88286 \text{ m}^2\cdot\text{Gauss},$$
$$a_1 = a_1' + \Delta a_1 = 0.02 - 0.00677 = 0.01323 \text{ m}\cdot\text{Gauss},$$
$$b_0' = b_0' + \Delta b_0 = -50 + 2.590 = -47.410 \text{ m}^2,$$
$$b_1 = b_1' + \Delta b'_1 = 0 - 2.562 = -2.562 \text{ m}.$$

The curve drawn in Fig. 25.1 corresponds to these parameter values. A comparison between Z_{calc} and Z_{obs} for the eight given datum points is given in the table below.

x	Z_{obs}	Z_{calc}	$Z_{\text{calc}} - Z_{\text{obs}}$
−40 m	70	23	−47 Gammas
−30	40	56	16
−20	150	156	6
−10	570	616	46
0	1 850	1 862	12
10	565	587	22
20	250	230	−20
40	110	81	−29

$\Sigma(Z_{\text{calc}} - Z_{\text{obs}}) = 6$ Gammas.

If the increments Δa_0, Δa_1, Δb_0, Δb_1 had been sufficiently small

$$\Sigma(Z_{calc} - Z_{obs})$$

would have been zero. Since this is not the case now

$$\Sigma(Z_{calc} - Z_{obs})^2$$

is not a minimum as required by a perfect equalizing calculation according to the method of least squares. The difference from zero of the first mentioned quantity is so small, however, that an improved equalizing calculation with better proximate values of $a'_0 \ldots b'_1$ would produce only very slight changes in the values obtained above for the parameters of the interpretation equation.

Since $\alpha = \beta = \gamma = 0$, the magnetic parameters of the sheet should be computed from the formulae (*6.5*).

Thus

$$x_0 = \tfrac{1}{2} b_1 = -1.28 \text{ m}, \qquad \varepsilon M_{//} = \frac{2a_0 + a_1 b_1}{2\sqrt{-4b_0 - b_1^2}} = 0.06447 \text{ m} \cdot \text{Gauss},$$

$$t_0 = \tfrac{1}{2}\sqrt{-4b_0 - b_1^2} = 6.72 \text{ m}, \qquad \varepsilon M_{\perp} = -\tfrac{1}{2} a_1 = -0.00662 \text{ m} \cdot \text{Gauss}.$$

II. *Simultaneous calculation of ΔZ and the magnetic parameters of the sheet.*

From the eight given datum points we choose five that have the greatest values of Z and put

$$\begin{aligned}
x_1 &= -20 \text{ m}, & Z_1 &= 150 \text{ Gammas} \\
x_2 &= -10 \text{ »} & Z_2 &= 570 \text{ »} \\
x_3 &= 0 & Z_3 &= 1\,850 \text{ »} \\
x_4 &= 10 \text{ »} & Z_4 &= 565 \text{ »} \\
x_5 &= 20 \text{ »} & Z_5 &= 250 \text{ »}
\end{aligned}$$

These datum points should according to the formulae (*12.3*) and (*12.4*) satisfy the equation

$$A_0 + A_1 x_i + A_2 x_i^2 + b_0 Z_i + b_1 x_i Z_i = x_i^2 Z_i,$$

where

$$\begin{aligned}
A_0 &= a_0 - b_0 \Delta Z, \\
A_1 &= a_1 - b_1 \Delta Z, \\
A_2 &= \Delta Z.
\end{aligned}$$

We calculate at first b_0 and b_1 according to (*16.9*). As

$$x_5 - x_4 = x_4 - x_3 = x_3 - x_2 = x_2 - x_1 = x_4$$

we obtain with the aid of the expressions (*19.3*)

$$c_Z^{(2)} = -2x_4^3(Z_1 - 3Z_2 + 3Z_3 - Z_4) = -6\,850 \cdot x_4^3,$$

$$c_{xZ}^{(2)} = -2x_4^3(x_1Z_1 - 3x_2Z_2 + 3x_3Z_3 - x_4Z_4) = -2x_4^4(-2Z_1 + 3Z_2 - Z_4)$$
$$= -1\,690 \cdot x_4^4,$$

$$c_{x^2Z}^{(2)} = -2x_4^3(x_1^2Z_1 - 3x_2^2Z_2 + 3x_3^2Z_3 - x_4^2Z_4) = -2x_4^5(4Z_1 - 3Z_2 - Z_4)$$
$$= 3\,350 \cdot x_4^5.$$

and

$d_Z^{(2)} = -2x_4^3(Z_2 - 3Z_3 + 3Z_4 - Z_5) = 7\,070 \cdot x_4^3,$

$d_{xZ}^{(2)} = -2x_4^3(x_2Z_2 - 3x_3Z_3 + 3x_4Z_4 - x_5Z_5) = -2x_4^4(-Z_2 + 3Z_4 - 2Z_5)$
$= -1\,250 \cdot x_4^4,$

$d_{x^2Z}^{(2)} = -2x_4^3(x_2^2Z_2 - 3x_3^2Z_3 + 3x_4^2Z_4 - x_5^2Z_5) = -2x_4^5(Z_2 + 3Z_4 - 4Z_5)$
$= -2\,530 \cdot x_4^5.$

The formulae (*16.9*) give

$$b_0 = -\frac{125 \cdot 335 - 169 \cdot 253}{685 \cdot 125 + 707 \cdot 169} x_4^2 = -0.41262 \cdot x_4^2 = -41.262 \text{ m}^2,$$

$$b_1 = \frac{685 \cdot 253 - 707 \cdot 335}{685 \cdot 125 + 707 \cdot 169} x_4 = -0.30979 \cdot x_4 = -3.0979 \text{ m}.$$

For the calculation of $A_2 = \Delta Z$ according to (*16.10*) we compute

$c^{(1)} = 2x_4^3,$

$c_F^{(1)} = x_4(Z_2 - 2Z_3 + Z_4) = -2\,565 \cdot x_4,$

$c_{xZ}^{(1)} = x_4(x_2Z_2 - 2x_3Z_3 + x_4Z_4) = x_4^2(-Z_2 + Z_4) = -5 \cdot x_4^2,$

$c_{x^2Z}^{(1)} = x_4(x_2^2Z_2 - 2x_3^2Z_3 + x_4^2Z_4) = x_4^3(Z_2 + Z_4) = 1\,135 \cdot x_4^3.$

The formula (*16.10*) gives

$$A_2 = \Delta Z = \frac{-b_0 c_Z^{(1)} - b_1 c_{xZ}^{(1)} + c_{x^2Z}^{(1)}}{c^{(1)}} = \frac{-0.41262 \cdot 2\,565 - 5 \cdot 0.30979 + 1\,135}{2} =$$
$$= 37.54 \text{ Gammas.}$$

Writing the interpretation equation

$$a_0 + a_1 x_i = (Z_i - \Delta Z)(x_i^2 - b_1 x_i - b_0)$$

we obtain for the datum points (3) and (4)

$a_0 = -b_0(Z_3 - \Delta Z) = 0.41262 \cdot 1\,812.5\ x_4^2 = 747.86 \cdot x_4^2 \text{ m}^2 \cdot \text{Gammas},$

$a_1 = (x_4 - b_1)(Z_4 - \Delta Z) + \frac{b_0}{x_4}(Z_3 - Z_4) = x_4(1.30979 \cdot 527.46 - 0.41262 \cdot$
$\cdot 1\,285) = 160.64\ x_4 = 1\,606.4 \text{ m} \cdot \text{Gammas}$

or $a_0 = 0.74786 \text{ m}^2 \cdot \text{Gauss},$
$a_1 = -0.01606 \text{ m} \cdot \text{Gauss}.$

Finally we obtain with the aid of formulae (*6.5*)

$x_0 = -1.55$ m, $\varepsilon M_{//} = 0.05798$ m · Gauss, $\Delta Z = 37.5$ Gammas.
$t_0 = 6.23$ » $\varepsilon M_{\perp} = -0.00803$ » »

The Z-profile corresponding to these values can obviously be calculated from the expression

$$Z_{\text{calc}} = \frac{a_0 + a_1 x}{-b_0 - b_1 x + x^2} + \Delta Z.$$

This profile curve passes through the five datum points used by the calculations above. The following values apply to the other three given datum points.

x	Z_{calc}	Z_{obs}	$Z_{calc}-Z_{obs}$
—40 m	44	70	—26 Gammas
—30 »	70	40	30 »
40 »	116	110	6 »

On comparing the results for the two cases I and II treated above it is apparent that the calculated Z-curve agrees much better with the given datum points in case II than in case I. In the latter case (I) the calculated Z-values are smaller than the observed values in the northernmost observation point and in the two southernmost ones, but greater than the observed Z-values in all the other points. This fact implies in itself that a better agreement can be obtained between Z_{calc} and Z_{obs} by supposing a regional anomaly ΔZ of convenient magnitude.

The upper edge of the ore-sheet is stated (19) to be situated just below the zero point of the x-axis at a depth of approximately 6 m below the datum plane (thickness of soil + height of stand). As there is no outcrop of the ore-sheet in the vicinity of the actual profile line, it is quite possible that these data may be a metre or two out. We can thus only deduce that the values given by I or II for x_0 and t_0 lie within the limits of uncertainty for the given data.

Calculations proceeding from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$.

For the earth's normal field we have

$$T = 0.478 \text{ Gauss}, \quad i = 66^\circ.5, \quad \vartheta = 135^\circ,$$

and according to (3.2)

$$T' = T\sqrt{1 - \cos^2 i \sin^2 \vartheta} = 0.459 \text{ Gauss}$$

$$\tan i' = -\frac{\tan i}{\cos \vartheta} = 3.253, \; i.\,e.\; i' = 73^\circ.$$

According to given data (19) the ore-sheet stands approximately vertically ($\varphi = 90^\circ$).

Measurements made by the authors on pieces of iron-ore taken from dumps gave a mean value of $\varkappa = 0.05$. Various circumstances, however, indicate that the average susceptibility of the ore-body may be substantially greater.

The thickness of the ore-sheet is not known but in all probability it does not exceed two metres.

We assume in the following that $N_{//} = 0$ and $N_{\perp} = 4\pi$.

a) On assuming that $\varkappa = 0.05$, we obtain according to (*8.4*) for cases I and II

I	II
$\tan (i'' - \varphi) = - 0.1672,$	$\tan (i'' - \varphi) = - 0.2255,$
$i'' - \varphi = - 9^\circ.5,$	$i'' - \varphi = - 12^\circ.7,$
$\varepsilon\varkappa T'' = 0.0653$ m · Gauss	$\varepsilon\varkappa T'' = 0.0595$ m · Gauss.

If we suppose that the ore-body has no remanent magnetization, *i. e.* $T'' = T'$ and $i'' = i'$ the values above give

$\varphi = i' + 9^\circ.5 = 82^\circ.5,$	$\varphi = i' + 12^\circ.7 = 85^\circ.7,$
$\varepsilon = 2.85$ m	$\varepsilon = 2.59$ m.

The calculated value of φ in I is approximately 8° smaller and that in II approximately 4° smaller than the given value. (The dotted line in Fig. *25.1* refers to I). The thickness is in both cases larger than is considered probable from available geological data.

b) Starting from the given value $\varphi = 90^\circ$ and the assumption that $i'' = i'$ we obtain from the formulae (*8.6*)

I	II
$\varkappa = 0.157,$	$\varkappa = 0.096,$
$\varepsilon\varkappa T'' = 0.0674$ m · Gauss,	$\varepsilon\varkappa T'' = 0.0606$ m · Gauss,

If we suppose that there is no remanent magnetization in the orebody *i. e.* $T'' = T'$ we then have

$\varepsilon = 0.94$ m,	$\varepsilon = 1.38$ m.

The above value of ($\varkappa$) in case I is thus approximately three times as large as, and in case II approximately twice as large as that found from determinations on samples. The values of the thickness are in both cases of a magnitude that can be considered probable. As the given $\varkappa$-value is very uncertain it is possible that the values of $\varkappa$ and ε in case II can be in good agreement with the real ones.

c) Finally we assume that the natural remanence is perpendicular to the plane of the ore-sheet. Further we use the given value $\varphi = 90^\circ$.

Putting $\psi' - \varphi = - 90^\circ$ in the equations (*3.1*) we may derive the formulae

$$\varepsilon\varkappa = \frac{\varepsilon M_{//}}{T' \cos (i' - \varphi)}, \quad k' = \sin (i' - \varphi) - \frac{\varepsilon M_{\perp}(1 + 4\pi\varkappa)}{\varepsilon\varkappa T'}.$$

Hence we obtain in the cases I and II

I	II
$\varepsilon\varkappa = 0.147,$	$\varepsilon\varkappa = 0.132,$
$k' = - 0.194 + 1.23 \cdot \varkappa,$	$k' = - 0.160 + 1.66 \cdot \varkappa.$

If $k' = 0$, *i. e.* no remanent magnetization exists, the above relations give the same values of $\varkappa$ and ε as in case b).

If $k' > 0$ we have in case (I) $\varkappa > 0.157$ and in case (II) $\varkappa > 0.096$, and if $k' < 0$ the $\varkappa$-values are smaller than the mentioned values.

A negative value of k' implies that $\psi' = 180°$. It is, however, not probable that the remanent cross magnetization should be in an opposite direction from the cross magnetization induced by the magnetic earth field.

For this reason we may infer that the susceptibility of the ore-body is at least as high as 0.1 and the thickness of the sheet not larger than 1.4 m. If the ore-sheet possesses a remanent magnetization then probably $\varkappa > 0.1$ and $\varepsilon < 1.4$ m.

§ 26. Spitzenberg I.

The continuous line of the Z-curve in Fig. *26.1* is taken from Fig. *26* of the previously mentioned paper (19). The positions of the observation points are not given.

The shape of the Z-curve (steep sides and a broad, flat-domed crown) is suggestive of an Z-anomaly that derives from one thick sheet or two (possibly more) thin sheets lying close together. Hence we shall attempt to reproduce the Z-profile by using an interpretation equation (*20.1*) of the form

$$A_0 + A_1x + A_2x^2 + A_3x^3 + B_0Z + B_1xZ + B_2x^2Z + B_3x^3Z = x^4Z.$$

For the calculations we choose the method of approach given in 3 b) of § 21. As calculation points we take those indicated by crossed circles in Fig. 26.1. Their coordinates are

$$x_e = x_1 = x_1' = 0, \qquad Z_e = Z_1 = Z_1' = 8\,300 \text{ Gammas}$$

$x_2 = -21$ m,	$Z_2 = 400$ »
$x_3 = -14$ »	$Z_3 = 2\,600$ »
$x_4 = -7$ »	$Z_4 = 7\,000$ »
$x_5 = 7$ »	$Z_5 = 7\,600$ »
$x_6 = 14$ »	$Z_6 = 5\,500$ »
$x_7 = 21$ »	$Z_7 = 3\,750$ »

Since $x_1 = x_1' = 0$ we have

$$U_i = \frac{Z_i}{Z_e - Z_i}\, x_i^2.$$

If we express all the abscissae as units of x_5 we obtain according to (*21.17*)

$$u_2 = \tfrac{1}{2}(U_2 - U_7) = -3.48094\, x_5^2, \qquad v_2 = \tfrac{1}{2}(U_2 + U_7) = 3.936640\, x_5^2,$$
$$u_3 = \tfrac{1}{2}(U_3 - U_6) = -3.016292\, x_5^2, \qquad v_3 = \tfrac{1}{2}(U_3 + U_6) = 4.840852\, x_5^2,$$
$$u_4 = \tfrac{1}{2}(U_4 - U_5) = -2.736264\, x_5^2, \qquad v_4 = \tfrac{1}{2}(U_4 + U_5) = 8.120879\, x_5^2.$$

As the values of v_2, v_3 and v_4 differ to a large extent from each other

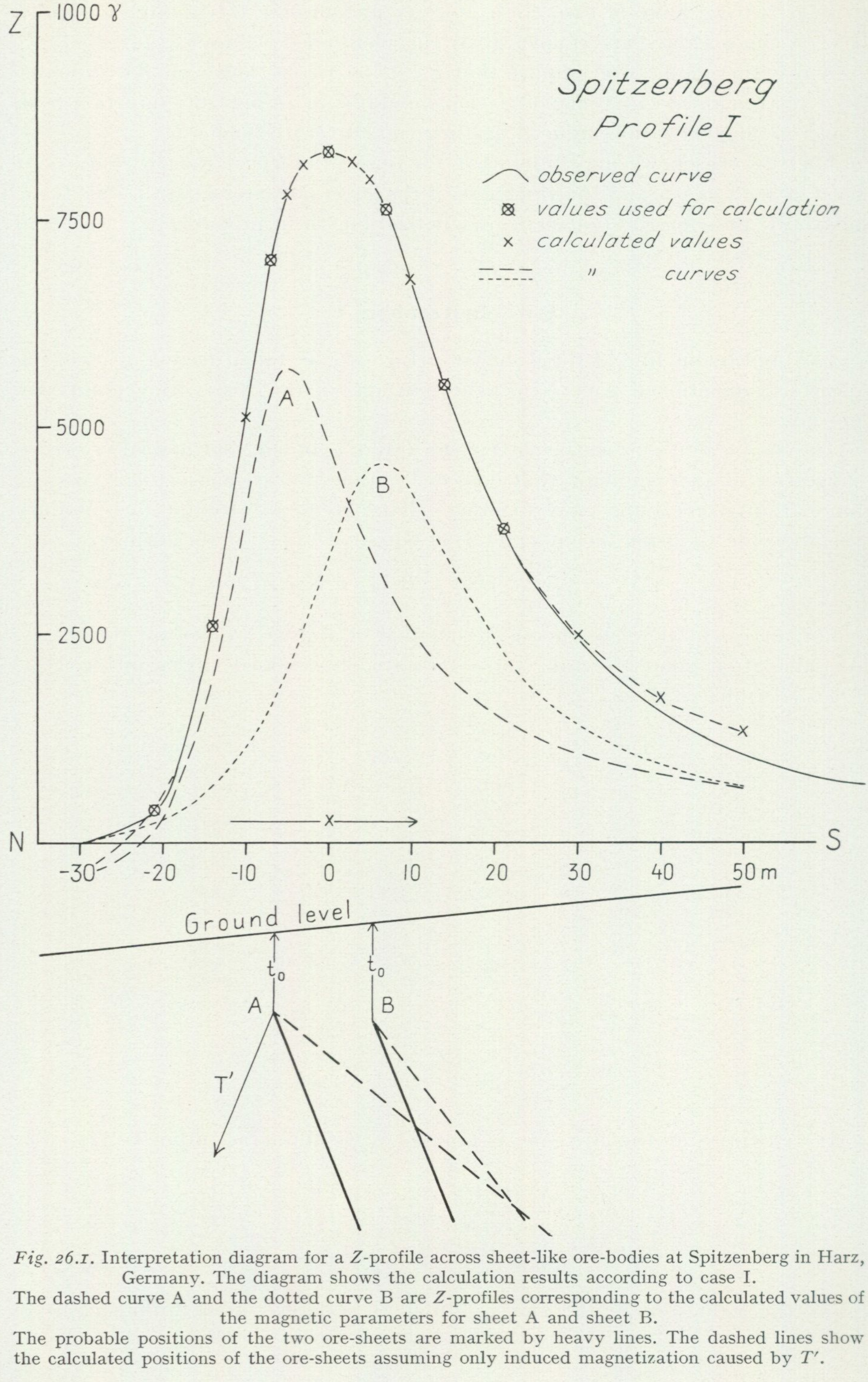

Fig. 26.1. Interpretation diagram for a Z-profile across sheet-like ore-bodies at Spitzenberg in Harz, Germany. The diagram shows the calculation results according to case I.
The dashed curve A and the dotted curve B are Z-profiles corresponding to the calculated values of the magnetic parameters for sheet A and sheet B.
The probable positions of the two ore-sheets are marked by heavy lines. The dashed lines show the calculated positions of the ore-sheets assuming only induced magnetization caused by T'.

the Z-profile can not emanate from *one sheet*. Hence we calculate according to (*21.19*)

$$c_1 = x_5(-u_2 + 4u_3 - 5u_4) = 5.09709 \cdot x_5^3,$$
$$c_2 = x_5^2(3v_2 - 8v_3 + 5v_4) = 13.68750 \cdot x_5^4,$$
$$c_3 = x_5^3(-9u_2 + 16u_3 - 5u_4) = -3.250894 \cdot x_5^5,$$
$$c_4 = x_5^4(27v_2 - 32v_3 + 5v_4) = -8.013577 \cdot x_5^6,$$

and further according to (*21.20*)

$$\alpha_1 = \frac{c_3^2 - c_2c_4}{c_2^2 - c_1c_3} = 0.58971 \cdot x_5^2, \quad \beta_1 = \frac{c_2c_3 - c_1c_4}{c_2^2 - c_1c_3} = -0.0179101 \cdot x_5.$$

Since $(-4\alpha_1 + \beta_1^2) < 0$, thus x_1'' and x_1''' become complex. This is apparently in agreement with the given Z-curve.

We now compute the quantities:

$$[xxxx]_1 = \alpha_1 x_1 x'_1 = 0, \qquad [xx]_1 = \alpha_1 + \beta_1(x_1 + x_1') + x_1x_1' = \alpha_1,$$
$$[xxx]_1 = \alpha_1(x_1 + x'_1) + \beta_1 x_1 x_1' = 0, \qquad [x]_1 = \beta_1 + x_1 + x_1' = \beta_1,$$

and (comp. *21.21*)

$$e_2 = x_5^2\left[\left(9 + \frac{\alpha_1}{x_5^2}\right)v_2 + 3\frac{\beta_1}{x_5}u_2\right] = 37.93828 \cdot x_5^4,$$

$$e_3 = x_5^2\left[\left(4 + \frac{\alpha_1}{x_5^2}\right)v_3 + 2\frac{\beta_1}{x_5}u_3\right] = 22.32615 \cdot x_5^4,$$

$$e_4 = x_5^2\left[\left(1 + \frac{\alpha_1}{x_5^2}\right)v_4 + \frac{\beta_1}{x_5}u_4\right] = 12.95885 \cdot x_5^4,$$

$$g_2 = x_5^2\left[\left(9 + \frac{\alpha_1}{x_5^2}\right)u_2 + 3\frac{\beta_1}{x_5}v_2\right] = -33.59272 \cdot x_5^4,$$

$$g_3 = x_5^2\left[\left(4 + \frac{\alpha_1}{x_5^2}\right)u_3 + 2\frac{\beta_1}{x_5}v_3\right] = -14.01730 \cdot x_5^4,$$

$$g_4 = x_5^2\left[\left(1 + \frac{\alpha_1}{x_5^2}\right)u_4 + \frac{\beta_1}{x_5}v_4\right] = -4.495311 \cdot x_5^4.$$

Finally we apply the formulae (*21.22*) and (*21.4*). We obtain:

$$A_0 = -B_0Z_e = \frac{4e_2 - 9e_3}{-5}Z_e = \frac{e_2 - 9e_4}{-8}Z_e = \frac{e_3 - 4e_4}{-3}Z_e = 9.83643\ x_5^4 Z_e,$$

$$A_1 = -B_1Z_e = \frac{-8g_2 + 27g_3}{-30\,x_5}Z_e = \frac{-g_2 + 27g_4}{-24\,x_5}Z_e = \frac{-g_3 + 8g_4}{-6\,x_5}Z_e$$
$$= 3.65752\ x_5^3 Z_e,$$

$$A_2 = -(B_2 + \alpha_1)Z_e = \frac{e_3 - e_2}{-5\,x_5^2}Z_e = \frac{e_4 - e_2}{-8\,x_5^2}Z_e = \frac{e_4 - e_3}{-3\,x_5^2}Z_e = 3.12243\ x_5^2 Z_e,$$

$$A_3 = -(B_3 - \beta_1)Z_e = \frac{-3g_3 + 2g_2}{-30\,x_5^3}Z_e = \frac{-3g_4 + g_2}{-24x_5^3}Z_e = \frac{-2g_4 + g_3}{-6\,x_5^3}Z_e$$
$$= 0.837782\ x_5 Z_e.$$

By calculating each of the parameters $A_0 \ldots A_3$ from three different expressions we obtain a check on all the preceding calculating operations.

The Z-curve corresponding to the parameters given above clearly passes through the calculation points and has the shape shown in Fig. *26.1* by the dashed curve. The points marked ($\times$) in this figure have been calculated from the expression

$$Z_{\text{calc}} = \frac{A_0 + A_1x + A_2x^2 + A_3x^3}{-B_0 - B_1x - B_2x^2 - B_3x^3 + x^4} =$$

$$= \left[1 + \frac{-\alpha_1x^2 + \beta_1x^3 - x^4}{-B_0 - B_1x - B_2x^2 - B_3x^3 + x^4}\right] Z_e.$$

The agreement between the given and the calculated curve is, as may be seen, very close within the area between the northernmost and southernmost calculation points. Farthest to the north the calculated curve falls somewhat below the given curve, and farthest to the south it passes somewhat above this curve. If we were to make Z_7 slightly less it is very likely that we should obtain better agreement between the calculated and observed curves in their southernmost parts without the agreement otherwise appreciably altering. On the other hand not much seems to be gained by changing Z_2 since the given Z-curve north of the calculating point (2) is probably disturbed.

For this reason we modify the results above in such a manner that the calculated Z-curve passes through the point

$$x_8 = 42 \text{ m}, \quad Z_8 = 1\,450 \text{ gammas},$$

on the given Z-curve instead of the calculation point (7) used above. By the following calculations we mark the quantities corresponding to the modified Z-curve with (′) and put

$$U_7' = U_7 + \Delta U_7.$$

It is easy to prove that

$$\alpha_1' = \alpha_1 + \Delta\alpha_1 = \frac{c_3^2 - c_2c_4 + \frac{3}{2}x_5^2(-c_4 + 6x_5c_3 - 9x_5^2c_2)\Delta U_7}{c_2^2 - c_1c_3 + \frac{1}{2}x_5(-c_3 + 6x_5c_2 - 9x_5^2c_1)\Delta U_7} =$$

$$= \frac{120.2541 \cdot x_5^2 - 202.0189 \cdot \Delta U_7}{203.9178 \cdot x_5^2 + 19.75104 \cdot \Delta U_7}\, x_5^2,$$

$$\beta_1' = \beta_1 + \Delta\beta_1 = \frac{c_2c_3 - c_1c_4 + \frac{1}{2}x_5(-c_4 + 3x_5c_3 + 9x_5^2c_2 - 27x_5^2c_1)\Delta U_7}{c_2^2 - c_1c_3 + \frac{1}{2}x_5(-c_3 + 6x_5c_2 - 9x_5^2c_1)\Delta U_7} =$$

$$= \frac{-3.650688 \cdot x_5^2 - 8.086518 \cdot \Delta U_7}{203.9178 \cdot x_5^2 + 19.75104 \cdot \Delta U_7}\, x_5.$$

Further, as $u_3' = u_3$, $u_4' = u_4$, $v_3' = v_3$, $v_4' = v_4$, we have

$$A_0' = A_0 + \Delta A_0 = \frac{e_3' - 4e_4'}{-3} Z_e = -\frac{x_5^2}{3}\left[4(v_3 - v_4) + \frac{\alpha_1'}{x_5^2}(v_3 - 4v_4) + 2\frac{\beta_1'}{x_5}(u_3 - 2u_4)\right] Z_e,$$

$$A_1' = A_1 + \Delta A_1 = \frac{-g_3' + 8g_4'}{-6x_5} Z_e = -\frac{x_5}{6}\left[-4(u_3 - 2u_4) - \frac{\alpha_1'}{x_5^2}(u_3 - 8u_4) - 2\frac{\beta_1'}{x_5}(v_3 - 4v_4)\right] Z_e,$$

$$A_2' = A_2 + \Delta A_2 = \frac{e_3' - e_4'}{3x_5^2} Z_e = \tfrac{1}{3}\left[4v_3 - v_4 + \frac{\alpha_1'}{x_5^2}(v_3 - v_4) + \frac{\beta_1'}{x_5}(2u_3 - u_4)\right] Z_e,$$

$$A_3' = A_3 + \Delta A_3 = \frac{g_3' - 2g_4'}{-6x_5^3} Z_e = -\frac{1}{6x_5}\left[4u_3 - 2u_4 + \frac{\alpha_1'}{x_5^2}(u_3 - 2u_4) + 2\frac{\beta_1'}{x_5}(v_3 - v_4)\right] Z_e.$$

Using the interpretation equation in the form (*21.12*) and inserting the above expressions of $A'_0 \ldots A'_3$ into this equation we obtain

$$\alpha_1'\left[-\frac{2(v_3 - 4v_4)}{x_5^2} + \frac{u_3 - 8u_4}{x_5^2}\left(\frac{x_i}{x_5}\right) + \frac{2(v_3 - v_4)}{x_5^2}\left(\frac{x_i}{x_5}\right)^2 - \frac{u_3 - 2u_4}{x_5^2}\left(\frac{x_i}{x_5}\right)^3 - \frac{6U_i'}{x_5^2}\right]$$
$$+ 2\beta_1' x_5\left[-\frac{2(u_3 - 2u_4)}{x_5^2} + \frac{v_3 - 4v_4}{x_5^2}\left(\frac{x_i}{x_5}\right) + \frac{2u_3 - u_4}{x_5^2}\left(\frac{x_i}{x_5}\right)^2 - \frac{v_3 - v_4}{x_5^2}\left(\frac{x_i}{x_5}\right)^3 + \frac{3U_i'}{x_5^2}\left(\frac{x_i}{x_5}\right)\right] = 2x_5^2\left[\frac{4(v_3 - v_4)}{x_5^2} - \frac{2(u_3 - 2u_4)}{x_5^2}\left(\frac{x_i}{x_5}\right) - \frac{4v_3 - v_4}{x_5^2}\left(\frac{x_i}{x_5}\right)^2 + \frac{2u_3 - u_4}{x_5^2}\left(\frac{x_i}{x_5}\right)^3 + \frac{3U_i'}{x_5^2}\left(\frac{x_i}{x_5}\right)^2\right].$$

Applying this equation on the calculation point (8) we obtain

$$-643.9033 \cdot \alpha_1' + 1\,112.435 \cdot x_5\beta_1' = -672.6475 \cdot x_5^2.$$

The above expression in connection with the former expressions of α_1' and β_1' give

$$\Delta U_7 = -0.4143147\, x_5^2, \quad \alpha_1' = 1.041989 \cdot x_5^2, \quad \beta_1' = -0.00153435 \cdot x_5^2.$$

Since $(-4\alpha_1' + \beta_1^{2\prime}) < 0$, the calculated Z-curve has no points of discontinuity.

Further we have

$$Z_7' = \frac{9x_5^2 Z_7 + (Z_e - Z_7)\Delta U_7}{9x_5^2 Z_e + (Z_e - Z_7)\Delta U_7} Z_e = 3\,632 \text{ gammas.}$$

$$\Delta Z_7 = Z_7' - Z_7 = -118 \text{ gammas.}$$

Thus the modified Z-curve which passes through the calculation point (8) gives a value Z_7' in the point x_7 at the reference line and this value is only 118 gammas (about 3 percent) less than the observed value Z_7.

Using the values of α_1' and β_1' we obtain

$$e_3' = 24.41678 \cdot x_5^4, \qquad g_3' = -15.22297 \cdot x_5^4,$$
$$e_4' = 16.58694 \cdot x_5^4, \qquad g_4' = -5.599881 \cdot x_5^4,$$

and

$$A_0' = -B_0'Z_e = 13.9770 \cdot x_5^4 Z_e,$$
$$A_1' = -B_1'Z_e = 4.92935 \cdot x_5^3 Z_e,$$
$$A_2' = -(B_2' + \alpha_1')Z_e = 2.60995 \cdot x_5^2 Z_e,$$
$$A_3' = -(B_3' - \beta_1')Z_e = 0.670534 \cdot x_5 Z_e.$$

Some comparisons between calculated Z-values corresponding to these parameters and observed Z-values are given in the table below.

x	Z_{calc}	Z_{obs}	$Z_{calc}-Z_{obs}$
—3.5 m	8 084	8 100	—16 gammas
3.5	8 147	8 150	—3 »
21	3 632	3 750	—118 »
28	2 512	2 650	—138 »
35	1 857	1 950	—93 »
56	990	850	140 »

From the table we may conclude that the calculated Z-curve runs a little below the observed Z-curve in the interval between $x_6 (= 14$ m) and $x_8 (= 42$ m) and runs a little above the given Z-curve north of x_8.

Calculation of the magnetic parameters of the single sheets.

It seems evident that the above determined values of $A_0' \,.\,.\, A_3'$, $B'_0 \,.\,.\, B_3'$ satisfy the given Z-curve better than the first determined values of $A_0 \,.\,.\, A_3$, $B_0 \,.\,.\, B_3$. In order to elucidate in what degree the results of interpretation depend on the accuracy with which the calculated profile curve coincides with the observed profile curve, we determine, however, in the following the magnetic parameters of the single sheets for the two mentioned groups of values. Thereby we speak of groups I and II, the latter corresponding to the dashed quantities.

As a first step we have to calculate the quantities p and q according to the expressions (*22.6*). Then we obtain

I	II
$A_0 = 9.83643\, x_5^4 Z_e,$	$A_0 = 13.9770\, x_5^4 Z_e,$
$A_1 = 3.65752\, x_5^3 Z_e,$	$A_1 = 4.92935\, x_5^3 Z_e,$
$A_2 = 3.12243\, x_5^2 Z_e,$	$A_2 = 2.60995\, x_5^2 Z_e,$
$A_3 = 0.837782\, x_5 Z_e,$	$A_3 = 0.670534\, x_5 Z_e,$

$B_0 = -9.83643\,x_5^4,$	$B_0 = -13.9770\,x_5^4,$
$B_1 = -3.65752\,x_5^3,$	$B_1 = -4.92935\,x_5^3,$
$B_2 = -3.71214\,x_5^2,$	$B_2 = -3.65194\,x_5^2,$
$B_3 = -0.855692\,x_5,$	$B_3 = -0.672068\,x_5,$
$p = -40.80934\,x_5^4,$	$p = -57.04070\,x_5^4,$
$q = -76.87496\,x_5^6,$	$q = -105.9286\,x_5^6,$
$Q < 0,$	$Q < 0,$
$B_2 + \frac{3}{8} B_3^2 < 0.$	$B_2 + \frac{3}{8} B_3^2 < 0.$

The two unequalities given nearest above (comp. *22.8* and *22.11*) imply according to the description in § 22 that the polynom

$$-B_0 - B_1 x - B_2 x^2 - B_3 x^3 + x^4$$

has only complex roots both in case I and II. This fact is, however, in these cases already clear as the values of α_1 and β_1 are complex.

Consequently we have to calculate $(b_{01} + b_{02})$ from the expression in (*22.14*). With the aid of the formulae (*22.13*) we obtain

$\cos\varphi = 0.7661218,$	$\cos\varphi = 0.6388337,$
$\varphi = 39°59'35''.8,$	$\varphi = 50°17'42''.3$
$b_{01} + b_{02} = -6.299217\,x_5^2,$	$b_{01} + b_{02} = -7.570936\,x_5^2,$

Further the formulae (*22.3*) and (*22.1*) give

$b_{01} = -2.860452\,x_5^2,$	$b_{01} = -3.191513\,x_5^2,$
$b_{02} = -3.438761\,x_5^2,$	$b_{02} = -4.379424\,x_5^2,$
$b_{11} = -2.092058\,x_5,$	$b_{11} = -2.343976\,x_5,$
$b_{12} = 1.236365\,x_5,$	$b_{12} = 1.671908\,x_5.$

and the formulae (*22.5*) give

$a_{01} = 1.661823\,x_5^2 Z_e,$	$a_{01} = 1.693390\,x_5^2 Z_e,$
$a_{02} = 1.440968\,x_5^2 Z_e,$	$a_{02} = 2.055741\,x_5^2 Z_e,$
$a_{11} = 0.5206821\,x_5 Z_e,$	$a_{11} = 0.6750455\,x_5 Z_e.$
$a_{12} = 0.3170999\,x_5 Z_e,$	$a_{12} = -0.0045115\,x_5 Z_e.$

As the above quantities are calculated with a high degree of accuracy it is possible to have a good check of the calculations by inserting the obtained values into the expressions (*20.3*) and (*20.4*) of $B_0 \ldots B_3$ respectively $A_0 \ldots A_3$.

With respect to the chosen coordinate system, the angles of strike and slope of the sheet-edges are zero, *i. e.* $\alpha = \gamma = 0$. The slope angle of the reference line corresponds to $\tan\beta = 0.093$. It is hence possible to calculate the magnetic parameters of the two sheets from the formulae (*24.1*). Noting the sheet corresponding to a_{01}, a_{11}, b_{01}, b_{11} with A, and the sheet corresponding to a_{02}, a_{12}, b_{02}, b_{12} with B, we obtain

I

Sheet A

$$x_0 = -0.922 \cdot x_5 = -6.5 \text{ m},$$
$$t_0 = 1.341 x_5 = 9.4 \text{ m},$$
$$\varepsilon M_{//} = 0.44191 \cdot x_5 Z_e = 0.2568 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = -0.22125 \cdot x_5 Z_e = -0.1301 \text{ m} \cdot \text{Gauss}$$
$$\varepsilon M_{\mathrm{T}} = 0.288 \text{ m} \cdot \text{Gauss}.$$

Sheet B

$$x_0 = 0.781 \cdot x_5 = 5.5 \text{ m},$$
$$t_0 = 1.763 \cdot x_5 = 12.3 \text{ m},$$
$$\varepsilon M_{//} = 0.48291 \cdot x_5 Z_e = 0.2840 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = -0.11506 \cdot x_5 Z_e = -0.0677 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\mathrm{T}} = 0.292 \text{ m} \cdot \text{Gauss}.$$

II

Sheet A

$$x_0 = -1.047 \cdot x_5 = -7.3 \text{ m},$$
$$t_0 = 1.360 \cdot x_5 = 9.5 \text{ m},$$
$$\varepsilon M_{//} = 0.36597 \cdot x_5 Z_e = 0.2126 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = -0.30641 \cdot x_5 Z_e = -0.1780 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\mathrm{T}} = 0.277 \text{ m} \cdot \text{Gauss}.$$

Sheet B

$$x_0 = 1.014 \cdot x_5 = 7.1 \text{ m},$$
$$t_0 = 1.935 \cdot x_5 = 13.5 \text{ m},$$
$$\varepsilon M_{//} = 0.53269 \cdot x_5 Z_e = 0.3095 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\perp} = 0.04748 \cdot x_5 Z_e = 0.0276 \text{ m} \cdot \text{Gauss},$$
$$\varepsilon M_{\mathrm{T}} = 0.311 \text{ m} \cdot \text{Gauss}.$$

Data of actual values of the ore-bodies are not available. It seems probable, however, that the deep angle (φ), the susceptibility ($\varkappa$), and the thickness of the ore-sheets are about the same as those for the case of Spitzenberg II.

A comparison between the results above in case I and case II shows that the values of x_0 as well as those of t_0 have differences of less than one meter for the sheet A and of a little more than one meter for the sheet B. From a practical point of view these differences are not essential.

The values of $\varepsilon M_{//}$, and especially of $\varepsilon M_{\perp}$, show significant differences for the two sheets A and B. This is still more the case as regards the ratio $\varepsilon M_{//} : \varepsilon M_{\perp}$. On the other hand, it is important to state that the values of $\varepsilon M_{\mathrm{T}}$ only differ a little (4 % for sheet A and about 7 % for sheet B). These facts obviously imply that results obtained by calculations from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ must be different for case I and case II as far as these quantities do not enter the used formulae in such a way that $\varepsilon M_{\mathrm{T}}$ can be substituted for the two mentioned quantities.

By the calculations in the foregoing we have assumed that the length of the sheets can be considered as infinite. Rössiger and Puzischa, however, state that the actual Z-profile is situated near the west end of a sheet-like ore-body. Consequently, the observed Z-curve ought to deviate perceptibly from the curve corresponding to an infinite length of the actual ore-body. The relative magnitude of this deviation must be greater the longer the distances are from the head points (or approximately from the maximum) on the terrain profile (reference line). For this reason the values of the magnetic parameters according to case I ought to be preferred to those according to case II, although the agreement between observed and calculated Z-curves are better in the latter case than in the former, if we consider the whole length of the given Z-curve (comp. p. 108).

In the following only case I is treated.

Calculations from $\varepsilon M_{//}$ *and* $\varepsilon M_{\perp}$.

The data of the normal earth's field are the same as for Spitzenberg II. Thus we have

$$T' = 0.475 \text{ Gauss}, \quad i' = 67^\circ.2.$$

On assuming that the sheets A and B only have induced magnetization and further that $N_{//} = 0$, $N_{\perp} = 4\pi$, and $\varkappa = 0.5$ we obtain according to (8.4)

	$\tan (i' - \varphi)$	φ	ε
Sheet A	-3.69	142°	4.14 m
Sheet B	-1.74	127°	2.40 »

According to the above calculations the sheets should have the positions marked by dashed lines in Fig. 26.1. As these calculation results are in bad agreement with the real facts the ore-sheets probably possess a remanent magnetization.

On assuming that the remanent magnetization is perpendicular to the plane of the ore-sheets as in the case of Spitzenberg II, and fixing $\varphi = 112^\circ$, we obtain with the aid of the relations (24.3) the following values of $\varepsilon\varkappa$ and $\varkappa N'_{\perp}$.

	$\varepsilon\varkappa$	$\varkappa N'_{\perp}$
Sheet A	0.762 m	0.960
Sheet B	0.843 »	3.17

Further, as the susceptibility probably has a value of about 0.5, we have for sheet A:

$$\varepsilon = 1.5 \text{ m} \qquad N'_{\perp} = 1.92 \approx 0.6\pi$$

and for sheet B:

$$\varepsilon = 1.7 \text{ m} \qquad N'_{\perp} = 6.34 \approx 2\pi.$$

§ 27. Benson Mines.

In the following we treat a magnetic aerial survey profile published by the Geological Survey of U. S. A. (24—25). The map (25) contains three magnetic profile curves measured in the same vertical plane, approximately at right angles to the direction of strike of the ore-body and taken at altitudes of 2 400, 5 400, and 11 400 feet above sea-level. The profile curves represent the anomaly for the total intensity of the earth's field, *i. e.* the difference between the observed and the normal intensities of the earth's field. Only the profile curve for the lowest line of flight (Fig. 27.1) approximately 1 000 feet above the ground, will be treated.

The highest value of the anomaly curve mentioned amounts only to circa 5 percent of the normal total intensity of the earth's field. It is thus probable

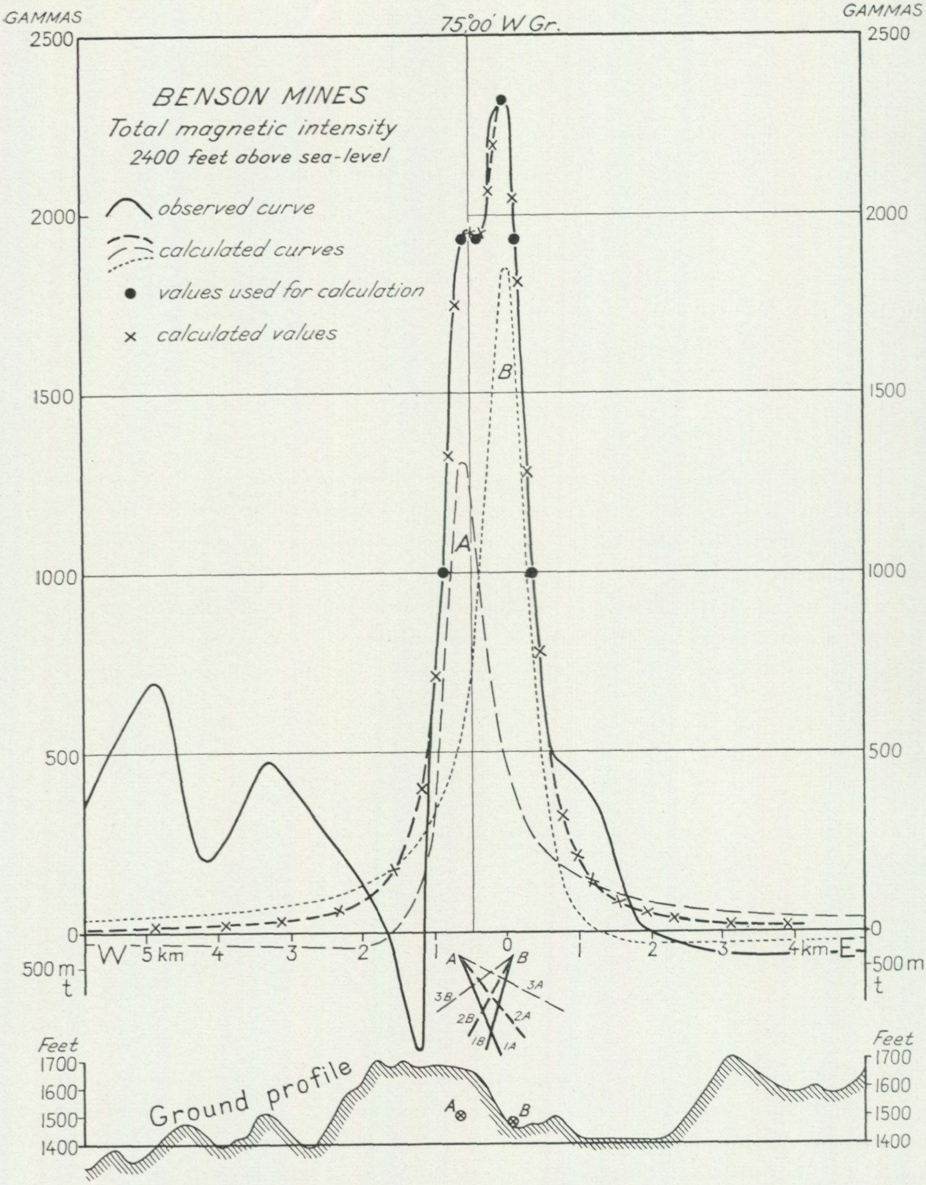

Fig. 27.1. Interpretation diagram for a profile curve showing the anomaly in the total intensity across Benson Mines in the Adirondac district, U. S. A. The profile curve is recorded by an airborne magnetometer 2 400 feet above sea-level (about 1 000 feet above ground).

The heavy dashed curve represents the calculated anomaly of the central situated ore-bodies. The fine dashed curve *A* and the dotted curve *B* correspond to the calculated values of the magnetic parameters of the sheet A and the sheet B. The positions of the two sheets calculated for three different alternatives (comp. p. 116), are marked.

(The length scale is incorrect. The distance marked as 1 km corresponds to only 800 m = $^1/_3$ mile.)

that the direction of the total field (the vector sum of normal earth field and disturbing field) varies only a few degrees along the measured line of flight. The observed anomaly curve drawn in Fig. *27.1* may hence be regarded as representing the intensity (F) of the anomalous field in the direction of the earth's normal field.

It is obvious from the shape of the given F-curve that the anomalous field originates from a number of disturbing bodies. The magnetic maps mentioned above show, however, that only the high peak in the middle of the given F-curve in Fig. *27.1* belongs to an anomalous band that has a persistent and reasonable extension on both sides of the actual profile line. Hence only this central anomaly can be subjected to treatment.

The flanks of this central anomaly is hidden by superimposed anomalous fields from extraneous disturbing bodies. The interpretation must hence be based wholly on the shape of the central peak of the F-curve. This peak is characterized by two maxima and one minimum, which imply the existence of at least two sheet-like bodies. We attempt as a first step to determine an interpretation equation of the form (*20.1*) that conforms satisfactorily with the actual pait of the given F-curve.

Calculation of the parameters of the interpretation equation.

We assume that the remainder field F_r is zero, which implies that the interpretation equation has the form given in § 26 p. 103.

Further we choose for the calculation the points marked in Fig. *27.1* by solid circles. These points constitute: 1) the minimum point (x_m, F_m) and the other two points having $F = F_m$, 2) the maximum point (x_M, F_M), 3) the two points (x_3, F_3), and (x_4, F_4) where $F_3 = F_4 = 1\,000$ gammas. It is thus possible to use the method of calculation treated in § 21 as a special case under 2 a (p. 74).

When determining the coordinates of the calculation points from the published original curve the abscissae have been measured in mm from the origin placed at x_M. In order to obtain the real distances of the points from x_M in meters, the values of the abscissae are to be multipled by a factor $k = 31.7$. The calculation points have the following coordinates.

$x_1 = -14.4 \cdot k$ m	$F_1 = F_m = 1\,931$ gammas
$x_m = x_1' = x_1'' = -9.5 \cdot k$ »	$F_m = F_1' = F_1'' =$ » »
$x_1''' = 4.3 \cdot k$ »	$F_1''' = F_m =$ » »
$x_M = x_2 = x_2' = 0$	$F_M = F_2 = F_2' = 2\,318$ »
$x_3 = -22.4 \cdot k$ »	$F_3 = 1\,000$ »
$x_4 = 10.0 \cdot k$ »	$F_4 =$ » »

We calculate at first (comp. *21.2*)

$$[x]_1 = x_1 + x_1' + x_1'' + x_1''' = -29.1\,k,$$

$$[xx]_1 = x_1x_1' + x_1x_1'' + x_1x_1''' + x_1'x_1'' + x_1'x_1''' + x_1''x_1''' = 220.23\,k^2,$$

$$[xxx]_1 = x_1x_1'x_1'' + x_1x_1'x_1''' + x_1x_1''x_1''' + x_1'x_1''x_1''' = 264.955\,k^3,$$

$$[xxxx]_1 = x_1x_1'x_1''x_1''' = -5\,588.28\,k^4.$$

From (*21.7*) we now obtain

$$C_3 = \frac{F_3}{F_m - F_3}\left[x_3^4 - x_3^3[x]_1 + x_3^2[xx]_1 - x_3[xxx]_1 + [xxxx]_1\right] = 38\,179.568 \cdot k^4,$$

$$C_4 = \frac{F_4}{F_m - F_4}\left[x_4^4 - x_4^3[x]_1 + x_4^2[xx]_1 - x_4[xxx]_1 + [xxxx]_1\right] = 56\,804.696 \cdot k^4.$$

Finally the formulae (*21.9*), where we put $F_1 = F_m$ and $F_2 = F_M$, and the formulae (*21.4*) give

$$A_0 = 33\,471.92 \cdot k^4 F_m, \qquad B_0 = -\,27\,883.64 \cdot k^4,$$
$$A_1 = 1\,586.991 \cdot k^3 F_m, \qquad B_1 = -\,1\,322.036 \cdot k^3,$$
$$A_2 = 76.35747 \cdot k^2 F_m, \qquad B_2 = -\,296.5875 \cdot k^2,$$
$$A_3 = 0.1728852 \cdot k F_m, \qquad B_3 = -\,28.92711 \cdot k.$$

The F-curve corresponding to the above values of the parameters is indicated by a heavy dashed line in Fig. *27.1*. The curve has been drawn by making use of the calculation points and those points marked by ($\times$) in the figure. These latter points have been calculated from the formula

$$F_{\text{calc}} = \frac{A_0 + A_1 x + A_2 x^2 + A_3 x^3}{-B_0 - B_1 x - B_2 x^2 - B_3 x^3 + x^4}.$$

As is obvious the agreement between the calculated and observed F-curves is very close at their central parts. The only noticeable deviation occurs here at their highest peaks, where the calculated peak is somewhat narrower than the observed one. A comparison between the flanks of the observed and calculated F-curves is of no use, as explained above.

Calculation of the magnetic parameters of the single sheets.

The calculations of a_{01}, a_{11}, b_{01}, b_{11} of the sheet (B) and of a_{02}, a_{12}, b_{02}, b_{12} of the other sheet (A) is completely analogous to those in the foregoing example (comp. pp. 109—110).

We obtain

$$p = 4B_0 + B_1 B_3 - \tfrac{1}{3} B_2^2 = -\,102\,613\,k^4,$$

$$q = B_1^2 - B_0 B_3^2 + B_2\left(\frac{p}{3} - 4B_0 + \frac{B_2^2}{27}\right) = 1\,178\,792\,k^6,$$

$$Q = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 < 0,$$

$$B_2 + \tfrac{3}{8} B_3^2 > 0,$$

$$p + B_2 B_3^2 + \tfrac{4}{3} B_2^2 + \tfrac{3}{16} B_3^4 < 0.$$

The three unequalities given nearest above (comp. *22.8* and *22.11*) imply according to the description in § 22 that the polynom

$$-B_0 - B_1 x - B_2 x^2 - B_3 x^3 + x^4$$

has only complex roots. Consequently, we have to calculate $(b_{01} + b_{02})$ from the expression in (*22.14*). The calculations give

$$(b_{01} + b_{02}) = - 424.7915\, k^2.$$

On inserting this value in the formulae (*22.3*) and (*22.1*) we obtain

$$b_{01} = - 81.1391\, k^2, \qquad b_{11} = 3.90488\, k,$$
$$b_{02} = - 343.6524\, k^2, \qquad b_{12} = - 32.83199\, k,$$

Further the formulae (*22.5*) give

$$a_{01} = 76.9928\, k^2F_m \qquad a_{11} = - 2.35169\, kF_m$$
$$a_{02} = 86.4338\, k^2F_m, \qquad a_{12} = 2.52458\, kF_m.$$

It is now possible to calculate the F-curve for each of the two sheets with the aid of the formulae

$$F_A = \frac{a_{02} + a_{12}x}{- b_{02} - b_{12}x + x^2}, \qquad F_B = \frac{a_{01} + a_{11}x}{- b_{01} - b_{11}x + x^2}.$$

These curves are in Fig. *27.1*. represented by a dashed line for F_A, and a dotted line for F_B.

The line of flight runs in an east-west direction; also we may put $\alpha = \beta = \gamma = 0$. The magnetic parameters of the sheets may thus be calculated from formulae (*6.7*). As in this case

$$v_x = \cos i \cdot \cos \vartheta, \quad v_z = \sin i, \quad i = 75^\circ \quad \text{and} \quad \vartheta = - 90^\circ$$

we have $v_x = 0$ and $v_z = 0.966$ and obtain the following values for the two sheets.

Sheet A.	Sheet B.
$x_0 = - 16.416 \cdot k = - 520$ m,	$x_0 = 1.952 \cdot k = 62$ m,
$t_0 = 8.612 \cdot k = 273$ m,	$t_0 = 8.794 \cdot k = 279$ m,
$\varepsilon M_{//} = 5.408 \cdot kF_m$	$\varepsilon M_{//} = 8.524 \cdot kF_m$
$= 3.310$ m · Gauss,	$= 5.218$ m · Gauss,
$\varepsilon M_{\perp} = - 1.307 \cdot kF_m$	$\varepsilon M_{\perp} = 1.217 \cdot kF_m$
$= - 0.800$ m · Gauss	$= 0.745$ m · Gauss.

The calculated positions of the magnetic edges of the sheets in relation to the ground surface are illustrated in the lower part of Fig. *27.1*. The shown ground profile, in which the height scale is ten times the horizontal scale, has been derived from the contour lines on a topographical map, having a scale of 1 : 62 500. The author has no detailed information as to the real positions of the ore-sheets.

Calculations from $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$.

From the data of the earth's normal field ($T = 0.58$ Gauss, $i = 75^\circ$ $\vartheta = - 90^\circ$) we obtain from (*3.2*)

$$T' = 0.56 \text{ Gauss}, \quad i' = 90^\circ.$$

Any data of the ore-body regarding dip angles, thickness or magnetization do not appear to have been published. For this reason only a few calculated results of dip angles and thickness for assumed values of $\varkappa$ are given below. Thereby it is assumed that the sheets only possess magnetization of induction and that $N_{//} = 0$ and $N_{\perp} = 4\pi$. The calculations are made with the aid of formulae (*8.4*).

Sheet A			Sheet B	
1. $\varkappa = 0.06$	$\varphi = 113^\circ$	$\varepsilon = 107$ m	$\varphi = 76^\circ$	$\varepsilon = 160$ m
2. $\varkappa = 0.2$	$\varphi = 130^\circ$	$\varepsilon = 39$ »	$\varphi = 63^\circ$	$\varepsilon = 52$ »
3. $\varkappa = 0.6$	$\varphi = 154^\circ$	$\varepsilon = 23$ »	$\varphi = 39^\circ$	$\varepsilon = 24$ »

The position of the sheets according to alternative 1, 2 and 3 are marked with heavy lines, heavy dotted lines, and thin dotted lines in the Fig. *27.1*.

§ 28. The Kursk anomaly.

In an article (8) by G. Gamburzeff and M. Polikarpoff, published in 1928 in Gerlands Beiträge für Geophysik, the authors describe an attempt at interpreting the gravitational and magnetic anomalies at the vast Russian iron ore deposit in the Kursk district.

They first treat the anomaly in the gravitational field, and make use of the observed data from a profile line (running at right angles to the direction of strike (N 30° W) of the ore body) in the neighbourhood of Stschigry. The results of four vertical bore holes along the said profile line indicate that the ore body consists of a sheet of ferriferous quartzite, 201 m in thickness, dipping 61° east, and having its upper edge located 160 m below the ground surface. From these data the authors find that the best agreement between the observed and calculated, anomalies is obtained, if the ore sheet is assumed to extend downwards to a vertical depth of 7 000 m from the ground surface, and the difference in specific gravity between the quartzite and the adjacent country rock is taken to be 0.85.

In order to study the magnetic field of the ore sheet, the authors make use of the data of the horizontal and vertical intensities from a profile line measured approximately 400 m south of the bore hole profile. The magnetic anomaly is assumed to originate from exactly the same disturbing masses as the gravitational anomaly, and these masses are assumed to have a homogeneous magnetizaton. Hence the authors make use of the relations, formulated by Eötvös, between the second derivatives of gravity potential, and the horizontal and vertical intensity components (X and Z) of the magnetic field in order to calculate the intensity (I) and the direction of the magnetization of the sheet. In this way they obtain I = 0.7 Gauss in a direction dipping 65° to the east, and use these values to interpret the X and Z observations (see Fig. 7, p. 229 of the paper referred to above). It appears that a considerable and systematic difference between the observed and the calculated profile curves of X and Z arises, if the values of the sheet dimensions obtained from the gravitational anomaly is used. By increasing the sheet dimensions and the depth 1.4 times, however, the authors arrive at a fairly good agreement between the observed and the calculated anomalous fields. The interpretation given by Gamburzeff and Polikarpoff is in accord with a sheet 280 m thick, dipping 61° to the east and having its upper and lower edges 224 m and 9 800 m respectively below the datum plane. This sheet has a magnetization intensity of I = 0.7 Gauss, the direction of which forms an angle of 65° downwards with the eastern direction of the profile line.

The above-mentioned observed data of the X- and Z-components are shown in Fig. *28.1*. The observed X-values are marked with open circles and the observed Z-values are marked with solid circles. The full lines drawn through these circles are in the following called the *observed X- and Z-curves*. These curves will be treated according to some of the interpretation methods that have been described earlier in this paper. Thereby is assumed that the ore-body can be considered as a magnetically homogeneous sheet, which extends to infinity on both sides of the actual profile line. Further, as data regarding the terrain contours not are available, we assume that the datum plane is horizontal.

I. The anomalous field is assumed to originate from a thin sheet of a very great extent in depth.

On knowing X and Z at two points (1 and 2) on a profile line, the magnetic parameters of the sheet may be calculated according to formulae (*6.9*), if the sheet has an infinite extent in depth. On choosing these points so that $Z_1 = Z_2$ the said formulae may be written

$$x_0 = \frac{x_1 X_1 - x_2 X_2}{X_1 - X_2}, \qquad 2\varepsilon M_{//} = -\, t_0 \frac{X_1 X_2 - Z_1^2}{Z_1},$$

$$t_0 = -\frac{(x_1 - x_2) Z_1}{X_1 - X_2}, \qquad 2\varepsilon M_{\perp} = -\, t_0 (X_1 + X_2).$$

We now pick out on the x-axis 7 pairs of points at which Z is 1.25, 1, 0.50, 0.25, 0.15, and 0.05 Gauss respectively. For each pair of points we note the data for x_1, x_2, X_1, X_2, and $Z_1 = Z_2$. These values are entered in the left-hand part of table 1 below. For each such group (1, 2 . . . 7) of observation data a series of values of x_0, t_0, $\varepsilon M_{//}$, and $\varepsilon M_{\perp}$ is calculated with the aid of the formulae above. Thereby we obtain the results entered in the right-hand part of the table.

From the table it is apparent that the values of the magnetic parameters of the sheet agree fairly well within groups 1—4, whereas for groups 5—7 agreement is poor. Here, for instance, the values of t_0 diminish quite appreciably. This is particularly marked in group 7 where t_0 amounts only to 170.6 m, whereas the mean value of t_0 for groups 1—4 is 282 m. Hence we exclude the values in group 7 by forming the mean value and furthermore, we assigne groups 1—4

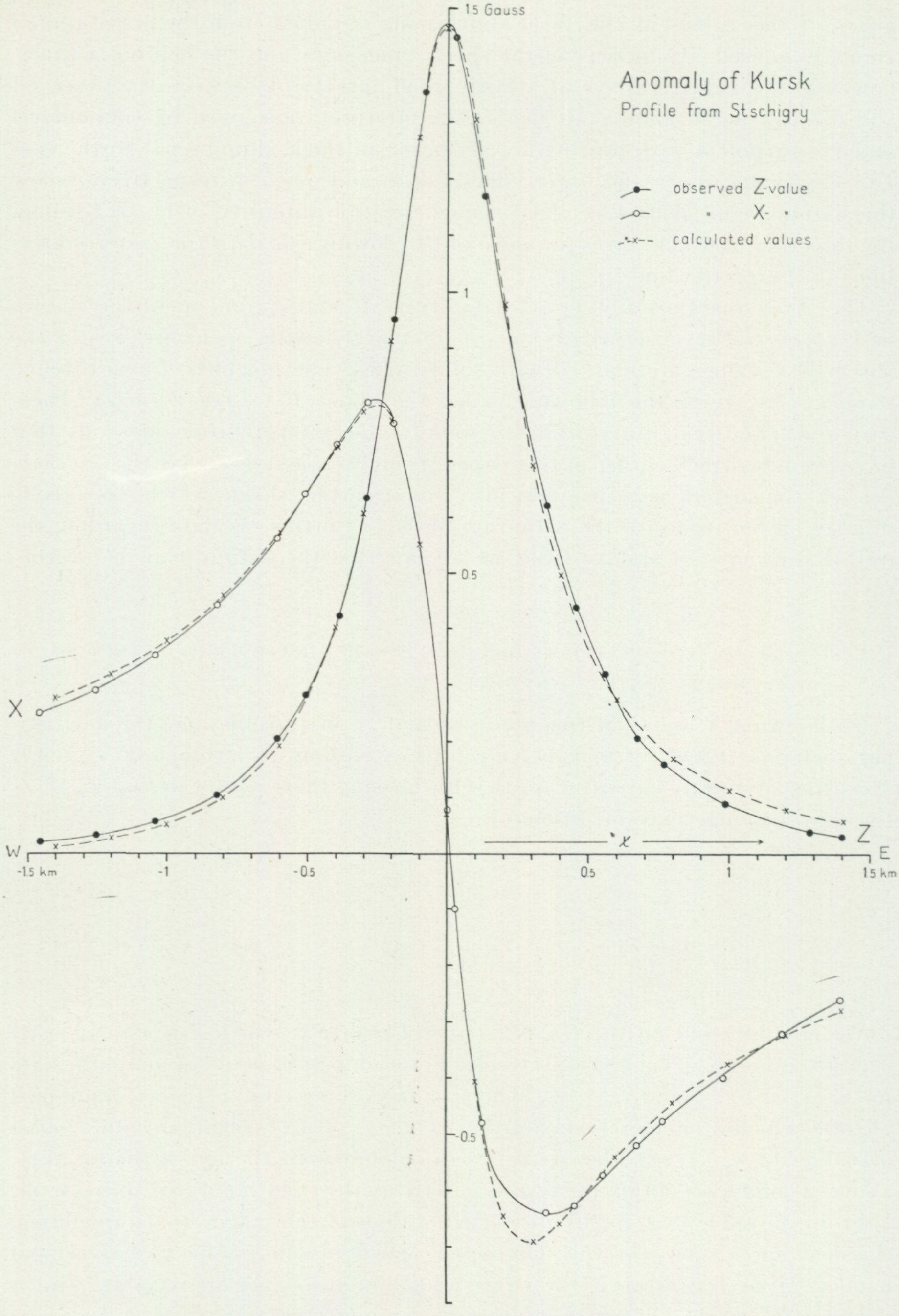

Fig. 28.1. Profile curves of X and Z across the Kursk anomaly in the vicinity of Stschigry. The dashed curves correspond to the calculated values in table 3.

Table 1.

Group	x_i m	X_i Gauss	Z_i Gauss	x_0 m	t_0 m	$\varepsilon M_{//}$ m · Gauss	$\varepsilon M_{\perp}$ m · Gauss
1	− 113 100	0.515 −0.405	1.25 1.25	−19.2	289.4	205.0	−15.9
2	− 180 192	0.740 −0.592	1.00 1.00	−14.6	279.2	200.8	−20.6
3	− 262 293	0.810 −0.635	0.75 0.75	−18.1	288.1	206.8	−25.2
4	− 343 417	0.766 −0.635	0.50 0.50	1.3	271.1	199.8	−17.8
5	− 540 610	0.612 −0.550	0.25 0.25	4.1	247.3	197.5	− 7.7
6	− 705 780	0.500 −0.472	0.15 0.15	16.5	229.1	197.5	− 3.1
7	−1 070 1 190	0.340 −0.322	0.05 0.05	30.2	170.6	191.8	− 1.5
Mean values for (1—7):				0	253.5	199.8	−13.1
» » » (1—4):				−12.6	282.0	203.1	−19.9

a double measure of importance as compared with groups 5—6. In this way we obtain the following values for the magnetic parameters of the sheet.

$x_0 = -8$ m, $t_0 = 273$ m, $\varepsilon M_{//} = 202$ m · Gauss, $\varepsilon M_{\perp} = -16$ m · Gauss.

The parameters of the interpretation equations for the X- and Z-curves are thus:

$$b_0 = -t_0^2 - x_0^2 = -74\,593 \text{ m}^2, \qquad b_1 = 2x_0 = -16 \text{ m},$$

and for the X-curve

$$a_0 = 2(x_0\varepsilon M_{//} - t_0\varepsilon M_{\perp}) = 6\,050 \text{ m}^2 \cdot \text{Gauss},$$
$$a_1 = -2\varepsilon M_{//} = -404 \text{ m} \cdot \text{Gauss},$$

and for the Z-curve

$$a_0 = 2(t_0\varepsilon M_{//} + x_0\varepsilon M_{\perp}) = 110\,564 \text{ m}^2 \cdot \text{Gauss},$$
$$a_1 = -2\varepsilon M_{\perp} = 34 \text{ m} \cdot \text{Gauss}.$$

On denoting the sheet by A and the corresponding X- and Z-components by X_A and Z_A we can write

$$X_A = \frac{a_0 + a_1 x}{-b_0 - b_1 x + x^2} = \frac{6\,050 - 404x}{74\,593 + 16x + x^2} \text{ Gauss},$$

$$Z_A = \frac{a_0 + a_1 x}{-b_0 - b_1 x + x^2} = \frac{110\,564 + 34x}{74\,593 + 16x + x^2} \text{ Gauss}.$$

The values of X_A and Z_A given in the table 2 below have been calculated according to these formulae.

Table 2.

x m	X_{obs} Gauss	X_A Gauss	$X_{obs} - X_A$ Gauss	Z_{obs} Gauss	Z_A Gauss	$Z_{obs} - Z_A$ Gauss
1 400	−0.262	−0.272	0.010	0.030	0.077	−0.047
1 200	−0.318	−0.312	−0.006	0.049	0.099	−0.050
1 000	−0.385	−0.365	−0.020	0.085	0.133	−0.048
800	−0.463	−0.436	−0.027	0.144	0.189	−0.045
600	−0.555	−0.532	−0.023	0.262	0.295	−0.033
400	−0.638	−0.646	0.008	0.530	0.515	0.015
300	−0.636	−0.680	0.044	0.715	0.713	0.002
200	−0.590	−0.635	0.045	0.960	0.996	−0.036
100	−0.400	−0.398	−0.002	1.260	1.322	−0.062
0	0.060	0.081	−0.021	1.470	1.482	−0.012
− 100	0.555	0.560	−0.005	1.285	1.291	−0.006
− 200	0.780	0.780	0	0.920	0.932	−0.012
− 300	0.793	0.796	−0.003	0.615	0.628	−0.013
− 400	0.724	0.735	−0.011	0.405	0.425	−0.020
− 600	0.565	0.585	−0.020	0.208	0.212	−0.004
− 800	0.450	0.469	−0.019	0.109	0.119	−0.010
−1 000	0.367	0.387	−0.020	0.062	0.072	−0.010
−1 200	0.302	0.328	−0.026	0.035	0.047	−0.012
−1 400	0.258	0.284	−0.026	0.021	0.031	−0.010

The remainder fields $(X_{obs} - X_A)$ and $(Z_{obs} - Z_A)$ given in table 2 are negative for the majority of the calculated points of the profile curves. This fact obviously implies that it is possible to have a better agreement between observed and calculated profile curves by assuming the occurrence of two additional anomalous fields X_B and Z_B which are constant, or linearly vary along the profile line from $x = -1\,400$ to $x = 1\,400$ m. In the point $x = 0$ on the profile line these additional fields ought to have the intensities

$$X_B = \frac{1}{2\,800} \int_{x=-1400}^{x=1400} (X_{obs} - X_A)\,dx \quad \text{and} \quad Z_B = \frac{1}{2\,800} \int_{x=-1400}^{x=1400} (Z_{obs} - Z_A)\,dx.$$

On evaluating the above integrals by using the values of $(X_{obs} - X_A)$ and $(Z_{obs} - Z_A)$ in table 2 and by assuming a linear rate of change in these data within the various intervals we obtain

$$X_B = -\,0.0098 \approx -\,0{,}010 \text{ Gauss}, \qquad Z_B = -\,0.0224 \approx -\,0{,}022 \text{ Gauss}.$$

We now assume that the anomalous fields X_B and Z_B arise from the fact that the ore-sheet has a finite extent in depth. From a magnetical point of view, this implies that X_B and Z_B can be considered as emanating from a sheet (B) having the upper edge coinciding with the lower edge of the ore-sheet and having a magnetization equal to that of sheet (A) in magnitude but in the opposite direction.

According to (6.8) the magnetic parameters of sheet (B) satisfy the relations

$$2\varepsilon M_{//} = x_0 X_B + t_0 Z_B,$$
$$2\varepsilon M_{\perp} = x_0 Z_B - t_0 X_B,$$

where $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ have the same numerical values as, but the opposite signs from the corresponding quantities for sheet (A). Hence we obtain for sheet (B)

$$x_0 = 2\frac{\varepsilon M_{//} X_B + \varepsilon M_{\perp} Z_B}{X_B^2 + Z_B^2} = 5\,637 \text{ m}, \quad t_0 = \frac{2\varepsilon M_{//} Z_B - \varepsilon M_{\perp} X_B}{X_B^2 + Z_B^2} = 15\,801 \text{ m}.$$

The calculation result above implies that the lower edge of the ore-sheet should be situated about 5 600 m east of the origin of the profile line, and at a depth of about 15 800 m under the datum plane. Further the ore-sheet should be dipping 70° to the east.

As the lower edge of the ore-sheet is situated at such a great depth under the datum plane, X_B and Z_B vary nearly linearly between $x = 1\,400$ m and $x = -1\,400$ m on the profile line. With the aid of the formulae (5.1, 5.2) we obtain for the former point $X_B = -0.0084$ Gauss, $Z_B = -0.0233$ Gauss, and for the latter point $X_B = -0.0113$ Gauss, $Z_B = -0.0205$ Gauss. Thus X_B alters by about 0.0001 Gauss per 100 m, and Z_B by -0.0001 Gauss per 100 m on moving from $x = -1\,400$ m to $x = 1\,400$ m.

The anomalous fields corresponding to the above-mentioned ore-sheet are obviously $X_{calc} = X_A + X_B$ and $Z_{calc} = Z_A + Z_B$. On adding the values of X_B and Z_B given above to the values of X_A and Z_B in table 2 we obtain the field data in table 3. These values are indicated in Fig. *28.1* by (×) and the resulting curves are dashed. As may be seen, agreement between the calculated and observed profile curves for X and Z are fairly satisfactory.

Table 3.

x m	X_{calc} Gauss	$X_{obs} - X_{calc}$ Gauss	Z_{calc} Gauss	$Z_{obs} - Z_{calc}$ Gauss	x m	X_{calc} Gauss	$X_{obs} - X_{calc}$ Gauss	Z_{calc} Gauss	$Z_{obs} - Z_{calc}$ Gauss
1 400	−0.280	0.018	0.054	−0.024	− 100	0.550	0.005	1.269	0.016
1 200	−0.321	0.003	0.076	−0.027	− 200	0.770	0.010	0.910	0.010
1 000	−0.374	−0.011	0.110	−0.025	− 300	0.786	0.007	0.606	0.009
800	−0.445	−0.018	0.166	−0.022	− 400	0.725	−0.001	0.403	0.002
600	−0.541	−0.014	0.272	−0.010	− 600	0.574	−0.009	0.191	0.017
400	−0.656	0.018	0.493	0.037	− 800	0.458	−0.008	0.098	0.011
300	−0.690	0.054	0.691	0.024	−1 000	0.376	−0.009	0.051	0.011
200	−0.645	0.055	0.974	−0.014	−1 200	0.317	−0.015	0.026	0.009
100	−0.408	0.008	1.300	−0.040	−1 400	0.273	−0.015	0.010	0.010
0	0.071	−0.011	1.460	0.010					

A principal objection can be made to the way used by the interpretation calculation above, viz.: the magnetic parameters of the sheet (A) are calculated by assuming $X_B = Z_B = 0$ (i. e. X_{obs} and Z_{obs} are used in the formulae instead of $X_{obs} - X_B$ and $Z_{obs} - Z_B$), and after that the values of X_B and Z_B are determined by using these values of the magnetic parameters.

At first it may be stated that the reason why, in the way above, we ought to obtain values of X_B and Z_B essentially differing from zero, is due to the fact that the values of the magnetic parameters from the groups 1—7 in table 1 have been given different measure of importance by determination of the ultimate mean values of the mentioned quantities. Further, it is obvious that if the quantities X_B and Z_B are omitted by calculating the magnetic parameters of the sheet (A) this fact influences the calculated values the less the ratios $X_B : X_{obs}$ and $Z_B : Z_{obs}$ are. Consequently, the parameter values are much less affected by the mentioned omission of X_B and Z_B within the groups 1—4, where both the mentioned ratios are small, than within the groups 5—7, where the ratio $Z_B : Z_{obs}$ is of the magnitude 0.1—0.5. These facts justify the way used by the determination of the ultimate mean values of the magnetic parameters and the values of X_B and Z_B.

II. The anomalous field is a priory supposed to originate from an ore-body of finite extent in depth.

We make use of the calculation method given as simple case in § 17. Thereby we imagine that the ore sheet limited in depth is replaced by two sheets (A and B) of infinite depth. The edge of the sheet (B), which is situated deeper, may be expected to lie so far from the measured part of the profile line that within this actual part the field components X_B and Z_B, originating from B, may be regarded as constant. Thus, we may calculate partly the parameters x_{01}, t_{01}, $\varepsilon M_{//}$, and $\varepsilon M_{\perp}$ of the sheet (A), and partly the quantities $\Delta X = X_B$ and $\Delta Z = Z_B$ with the aid of formulae (17.1—17.2). Further we obtain the parameters x_{02} and t_{02} of sheet (B) from the formulae

$$x_{02} = -2\,\frac{\varepsilon M_{//}\cdot \Delta X + \varepsilon M_{\perp}\cdot \Delta Z}{(\Delta X)^2 + (\Delta Z)^2}, \qquad t_{02} = 2\,\frac{\varepsilon M_{\perp}\cdot \Delta X - \varepsilon M_{//}\cdot \Delta Z}{(\Delta X)^2 + (\Delta Z)^2},$$

where $\varepsilon M_{//}$ and $\varepsilon M_{\perp}$ have the values valid for sheet (A).

Table 4.

Group	x m	X_i Gauss	Z_i Gauss	Group	x m	X_i Gauss	Z_i Gauss
1	300	− 0.636	0.715	5	− 100	0.555	1.285
	1 400	− 0.262	0.030		600	− 0.555	0.262
	400	0.724	0.405		− 1 000	0.367	0.062
2	200	− 0.590	0.960	6	− 200	0.780	0.920
	1 200	− 0.318	0.049		500	− 0.605	0.380
	− 500	0.640	0.280		− 1 200	0.302	0.035
3	100	− 0.400	1.260	7	− 300	0.793	0.615
	1 000	− 0.385	0.085		400	− 0.638	0.530
	− 600	0.565	0.208		− 1 400	0.258	0.020
4	0	0.060	1.470				
	800	− 0.463	0.144				
	− 800	0.450	0.109				

In order to increase the prospects of good agreement between calculated and observed profile curves of X and Z, it is advisable to carry out calculations from a great many point groups. In table 4 there are denoted the values of x_i, X_i, and Z_i from 7 suitable selected point groups, numbered from 1 to 7. Within each such group in the table we let the values of the uppermost, middle, and lowest lines correspond to $i = 1$, 2 and 3 respectively.

When carrying out the interpretation calculations one has first, for each group of given data in table 4, to calculate h_i, l_i, m_i, and n_i corresponding to $i = 2$ and 3 according to the formulae (17.1). After that the quantities ΔX, ΔZ, x_{01}, t_{01}, $\varepsilon M_{//}$, and $\varepsilon M_{\perp}$ may be determined at once with the aid of the formulae (17.2). The calculated results of the different groups are denoted in table 5.

Table 5.

Group	ΔX Gauss	ΔZ Gauss	x_{01} m	t_{01} m	$\varepsilon M_{//}$ m · Gauss	$\varepsilon M_{\perp}$ m · Gauss
1	0.0262	— 0.0491	8.72	300.1	210.9	— 8.8
2	0.0029	— 0.0502	0.60	297.3	208.5	— 12.1
3	— 0.0077	— 0.0416	— 1.10	288.9	207.1	— 8.9
4	— 0.0128	— 0.0284	1,25	272.6	204.2	— 7.2
5	— 0.0156	— 0.0163	— 4.45	264.1	199.5	— 12.3
6	— 0.0284	— 0.0035	— 14.51	268.5	199.8	— 21.3
7	— 0.0291	0.0022	— 18.11	271.7	198.7	— 26.3
Mean:	— 0.0092	— 0.0267	— 3.94	280.5	204.1	— 13.8

On using the mean values $\Delta X = -0.009$ Gauss, $\Delta Z = -0.027$ Gauss, $\varepsilon M_{//} = 204$ m · Gauss, $\varepsilon M_{\perp} = -14$ m · Gauss, we obtain

$$x_{02} \approx 3\,600 \text{ m}, \qquad t_{02} \approx 13\,900 \text{ m}.$$

The magnetic parameters of the sheets (A) and (B) thus become

Sheet A	Sheet B
$x_{01} = -4$ m,	$x_{02} = 3\,600$ m,
$t_{01} = 280$ m,	$t_{02} = 13\,900$ m,
$\varepsilon M_{//} = 204$ m · Gauss,	$\varepsilon M_{//} = -204$ m · Gauss,
$\varepsilon M_{\perp} = -14$ m · Gauss,	$\varepsilon M_{\perp} = 14$ m · Gauss.

III. Graphical interpretation.

In the XZ-diagram in Fig. 28.2 the thick arc-line represents the *relation figure* of the observed profile curves of X and Z according to the description in § 6 p. 24. If we start from the point $x = -1\,400$ m on the ground profile, and follow this line towards the point near $x = 0$ where the observed X-value is 0, that part of the above-mentioned arc-line is described which lies to the left of the Z-axis in the diagram. The (×)-signs, from the bottom upwards, correspond to the points $x = -1\,400$, $-1\,200$, $-1\,000$, -800, -600, -500, -400, -300, -200, -100, and 0 m on the profile line. To the right of the Z-axis the

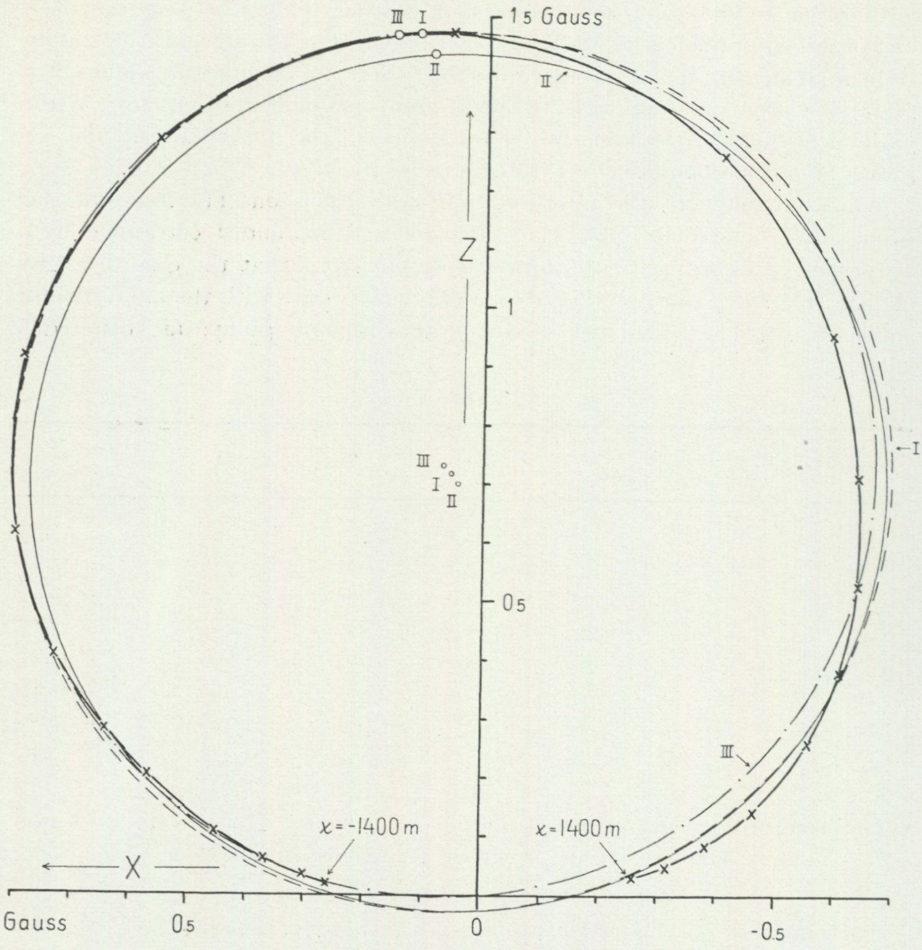

Fig. 28.2. Relation figures corresponding to observed and calculated profile curves of X and Z according to Fig. 28.1 and calculations under I, II and III.

The heavy line represents the *observed relation figure.*

The small open circles in the centre are the centres of the calculated *relation circles* and the small open circles at the top of the figure correspond to the head points in the cases I, II and III.

(×)-signs, from the top downwards, correspond to $x = 100$, 200, 300, 400, 500, 600, 800, 1 000, 1 200, and 1 400 m on the profile line. The X-axis has been taken positive to the left of the XZ-diagram in order to facilitate a comparison with the interpretation diagram in Fig. 28.1.

The left half of the above-mentioned arc-line in Fig. 28.2 agrees very well with a circular line, whereas the right half of the arc-line has a shape differing quite considerably from a circular line. On the other hand, we know (comp. § 6 p. 24) that the observed relation figure must be a circle in this actual case, if the profile curves of X and Z only emanate from an ore-sheet of very great extension in depth (ΔX and ΔZ constante). For this reason the shape of the observed arc-line invites us to base the interpretation on a circle coinciding as closely as possible with the left half of the thick arc-line in Fig. 28.2. The lacking agreement between this circle and the right half of the »observed» arc-line is thereby assumed to originate from strange disturbing bodies.

If a circle (III) is drawn through the three (×)-signs that correspond to $x = 0$, — 200, and — 1 400 m, it will follow the left part of the arc-line very closely (comp. Fig. 28.2). By a graphical construction of this circle, the coordinates a and b of the centre, and the radius r are determined. These quantities (comp. the formulae 6.11) are functions of $\frac{\varepsilon M_{\parallel}}{t_0}$, $\frac{\varepsilon M_{\perp}}{t_0}$, ΔX, and ΔZ, but none of the latter quantities may be obtained solely from the values of a, b and r. Consequently, the magnetic parameters of the sheet can not be determined only by a construction of the *relation circle*. In order to determine the magnetic parameters, the way of calculation already used under II is the most suitable.

As regards the quantities ΔX and ΔZ, at least one of these must be differing from zero, if the relation circle does not pass through the origo of the XZ-diagram. On the other hand, ΔX and ΔZ are not necessarily zero, even though the relation circle passes through the mentioned origo.

Although the observed relation figure can not be used for the proper determination of the magnetic parameters of the sheet, it may give a good and convenient guidance by choice of expedient alternatives of interpretation, by choosing suitable calculation points and by assessing various results of interpretation. It should be pointed out, however, that a good agreement between an *observed* arc-line and a *calculated* circle does not necessarily imply an equally good agreement between *observed* and *calculated* X- and Z-curves. The fact is, that the arc-line represents a function of X and Z, whereas the profile curves of X and Z are functions of x.

For the three above-mentioned points of circle III we have

x_i m	X_i Gauss	Z_i Gauss	h_i	l_i	m_i	n_i
0	0.060	1.470				
— 200	0.780	0.920	−13.5461	265.903	−175.417	133.999
— 1 400	0.260	0.020	−15.0293	248.961	−130.688	947.491

From this we obtain:

$\Delta X = -0.0209$ Gauss, $x_{01} = -17.1$ m, $\varepsilon M_{\parallel} = 197.0$ m · Gauss,
$\Delta Z = -0.0007$ » , $t_{01} = 268.8$ », $\varepsilon M_{\perp} = -23.4$ » · » .

A comparison between the observed X- and Z-curves and the ones calculated from the above parameter values, are given in table 6 below.

Table 6.

x m	$X_{obs}-X_{calc}$ Gauss	$Z_{obs}-Z_{calc}$ Gauss	x m	$X_{obs}-X_{calc}$ Gauss	$Z_{obs}-Z_{calc}$ Gauss
− 1 400	0	0	100	0.011	− 0.035
− 1 200	− 0.002	0.001	200	0.042	− 0.011
− 1 000	0.003	0.005	300	0.035	0.017
− 800	0.002	0.009	400	0.001	0.021
− 600	− 0.002	0.018	500	− 0.021	− 0.002
− 500	− 0.003	0.008	600	− 0.025	− 0.035
− 400	− 0.002	0.004	800	− 0.024	− 0.050
− 300	− 0.001	0.007	1 000	− 0.013	− 0.053
− 200	0	0	1 200	0.003	− 0.055
− 100	0.004	− 0.004	1 400	0.021	− 0.052
0	0	0			

IV. The anomalous field is assumed to originate from a thick sheet.

On determining the derivatives of X and Z with respect to x from the observed profile curves, we may draw the *relation figure* for $a_{X'}$, and $a_{Z'}$, according to the description on page 88. This *relation figure* will, however, deviate essentially from the form of a parabola, especially as regards the part corresponding to the east halves of the X- and Z-curves. Thus, carrying out the interpretation calculations according to the method developed for a thick sheet (comp. pp. 86—88), the agreement between the observed and calculated profile curves (considered in their whole length) will probably not be essentially better than that obtained in the cases I and III. On the other hand we may use the results from the cases treated above in order to carry out the interpretation in the case of a thick sheet.

In the following we use the same symbols as under III for a sheet of thickness B and mark the corresponding quantities for a thin sheet of thickness ε with a dash ($'$). Thereby x_{01} and x_{02} denote the x-coordinates of the centre line of the upper and the lower edge-side of the thick sheet.

We stipulate that the value X_0 of X and the value Z_0 of Z in the point $x = x_{01} = x_{01}' \approx 0$ each should be equal in the case of a thin and a thick sheet and that the same is valid for the expressions

$$\int_{-x_a}^{x_a} X_{calc}\, dx \text{ and } \int_{-x_a}^{x_a} Z_{calc}\, dx.$$

If x_a is much greater than t_{01}, t'_{01} and $B : \sin \varphi$ and if further X_B and Z_B vary approximately linearly in the interval $-x_a < x < x_a$, the above conditions may be written

$$\left.\begin{aligned}
&-4M_\perp \sin\varphi\cdot\tan^{-1}\left(\frac{B}{2t_{01}\sin\varphi}\right)+X_B=-\frac{2\varepsilon M_\perp'}{t_{01}'}+X_B'=X_0,\\
&4M_{/\!/}\sin\varphi\cdot\tan^{-1}\left(\frac{B}{2t_{01}\sin\varphi}\right)+Z_B=\frac{2\varepsilon M_{/\!/}'}{t_{01}'}+Z_B'=Z_0,\\
&-2BM_\perp\tan^{-1}\frac{x_a}{t_{01}}+x_aX_B=-2\varepsilon M_\perp'\tan^{-1}\frac{x_a}{t_{01}'}+x_aX_B',\\
&2BM_{/\!/}\tan^{-1}\frac{x_a}{t_{01}}+x_aZ_B=2\varepsilon M_{/\!/}'\tan^{-1}\frac{x_a}{t_{01}'}+x_aZ_B',
\end{aligned}\right\}\quad\ldots\quad(28.1)$$

where X_B, X_B', Z_B, Z_B' refer to the point $x = 0$. From these equations we may derive the relations

$$BM_\perp : BM_{/\!/} = \varepsilon M_\perp' : \varepsilon M_{/\!/}' = k,\quad X_B + kZ_B = X_B' + kZ_B'.$$

Further the relation

$\tan\varphi = \frac{t_{02}-t_{01}}{x_{01}-x_{02}} \approx \frac{t_{02}}{x_{01}-x_{02}}$ and the equations (6.8), applied to the sheet (B) give: $X_B(\tan\varphi - k) + Z_B(1 + k\tan\varphi)$.

Hence

$$X_B = \frac{1+k\tan\varphi}{1+k^2}(X_B'+kZ_B'),\quad Z_B = \frac{k-\tan\varphi}{1+k^2}(X_B'+kZ_B')\quad\ldots\quad(28.2)$$

If $B : 2t_{01}\sin\varphi < 1$, the second equation in (28.1) may be written

$$2BM_{/\!/}\left[1-\frac{1}{3}\left(\frac{B}{2t_{01}\sin\varphi}\right)^2+\frac{1}{5}\left(\frac{B}{2t_{01}\sin\varphi}\right)^4-\frac{1}{7}\left(\frac{B}{2t_{01}\sin\varphi}\right)^6\ldots\right]=t_{01}(Z_0-Z_B).$$

On using the four first terms of the series expansion, the above equation may be written in the form

$$\left[\left(\frac{B}{2t_{01}\sin\varphi}\right)^2-\frac{7}{15}\right]^3+p\left[\left(\frac{B}{2t_{01}\sin\varphi}\right)^2-\frac{7}{15}\right]+q=0,$$

$$\text{where } p=\frac{126}{75}=1.68,\qquad q=7\left[\frac{t_{01}(Z_0-Z_P)}{2BM_{/\!/}}+\frac{7\cdot 61}{(15)^3}-1\right]$$

$$=\frac{7t_{01}(Z_0-Z_B)}{2BM_{/\!/}}-6.1144.$$

As p is positive we obtain (comp. p. 81 case a))

$$\left.\begin{aligned}
&B=\overset{+}{(-)}\,2t_{01}\sin\varphi\sqrt{\frac{7}{15}+u-\frac{p}{3u}},\\
&\text{where } u=\left[-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}\,\right]^{1/3}.
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(28.3)$$

The above formula gives useful values of B only if $-1.0477 < q < \frac{7 \cdot 7 \cdot 61}{(15)^3} =$
$= 0.88563$.

The lower limit corresponds to $B : 2t_{01} \sin \varphi = 1$ and the higher limit corresponds to $B = 0$.

The formulae above have been applied to the interpretation results from I and III. As the dip of the ore-sheet is 61° to the east ($\varphi = 119°$) we obtain with the aid of formulae (28.2) in case

$$\text{I:}\quad X_B = -0.00912 \text{ Gauss}, \quad Z_B = -0.01376 \text{ Gauss},$$
$$\text{III:}\quad X_B = -0.02492 \text{ Gauss}, \quad Z_B = -0.03459 \text{ Gauss}.$$

On inserting these values in the fourth equation in (28.1) and putting $x_a = 1\,400$ m, we may calculate $BM_{//}$ for different values of t_{01} and compute the corresponding values of t_{02} in the way shown under I and III. Further, we obtain B from (28.3). After that $M_T = M_{//} \sqrt{1 + k^2}$ has been calculated. The results are given in table 7.

Table 7.

t_{01} m	I			III		
	t_{02} m	B m	M_T Gauss	t_{02} m	B m	M_T Gauss
200	20 170	320	0.597	8 550	346	0.603
220	20 370	296	0.653	8 640	340	0.619
230	20 470	271	0.717	8 680	329	0.644
240	20 580	236	0.825	8 720	310	0.687
250	20 680	192	1.023	8 770	282	0.757
260	20 780	129	1.523	8 810	246	0.871

The intensity of the magnetization (M_1) of the ore-body has been determined to 0.7 Gauss (comp. p. 116). Nearly the same value is according to table 7 valid in case I for $t_{01} = 230$ m and in case III for $t_{01} = 240$ m. For $M_T = 0.7$ we obtain more precisely by graphical interpolation in case I: $t_{01} = 228$ m, $t_{02} = 20\,450$ m, $B = 277$ m, and in case III: $t_{01} = 242$ m, $t_{02} = 8\,730$ m, $B = 305$ m. Further M_T forms an angle of 65.5° downwards with the eastern direction of the profile line in the former case and an angle of 68° in the latter case.

The values mentioned above for case I agree very well with the values obtained by Gamburzeff and Polikarpoff, except the value of t_{02} which is about twice that given by the mentioned authors. In case III the value of t_{02} is in fairly good agreement with that of the russian authors but other values are somewhat greater. It is, however, apparent that the determination of t_{02} is much uncertain.

Finally may be pointed out that the agreement between observed and calculated values of X and Z in the points $x = 1\,400$ m, and $= x - 1\,400$ m for the values mentioned above of case I is better than that given in table 3 and also better than that valid for the mentioned values of case III.

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Table 1.

$$l = \infty \quad X = -2\varepsilon'[k'_x M_{//} + k'_t M_{\perp}], \quad (4.1\text{ a}) \qquad Z = 2\varepsilon'[k'_t M_{//} - k'_x M_{\perp}]. \quad (4.1\text{ b})$$

Table 2.

$$l \text{ finite}, \; t > 0 \quad \begin{cases} X = -2\varepsilon' \dfrac{l'}{\sqrt{1 + l'^2}}[k'_x M_{//} + (k'_t - C'_{\perp} \sin\varphi) M_{\perp}], \quad \ldots\ldots\ldots \quad (4.3\text{ a}) \\ Z = 2\varepsilon' \dfrac{l'}{\sqrt{1 + l'^2}}[k'_t M_{//} - (k'_x - C'_{\perp} \cos\varphi) M_{\perp}]. \quad \ldots\ldots\ldots \quad (4.3\text{ b}) \end{cases}$$

Table 3.

$$l \text{ finite}, \; t = 0 \quad \begin{cases} X = -2\varepsilon''[k''_x M_{//} - C''_{\perp} \sin\varphi M_{\perp}], \quad \ldots\ldots\ldots \quad (4.4\text{ a}) \\ Z = -2\varepsilon''[k''_x - C''_{\perp} \cos\varphi] M_{\perp}. \quad \ldots\ldots\ldots \quad (4.4\text{ b}) \end{cases}$$

Table 1.

$l = \infty$

$\overset{+}{(-)}\, x'$	k'_t	$\overset{+}{(-)}\, k'_x$	$\overset{+}{(-)}\, x'$	k'_t	$\overset{+}{(-)}\, k'_x$	$\overset{+}{(-)}\, x'$	k'_t	$\overset{+}{(-)}\, k'_x$	$\overset{+}{(-)}\, x'$	k'_t	$\overset{+}{(-)}\, k'_x$
0.0	1.0000	0	3.0	0.1000	0.3000	6.0	0.0270	0.1621	9.0	0.0122	0.1098
0.1	0.9901	0.0990	3.1	0943	2922	6.1	0262	1596	9.1	0119	1086
0.2	9615	1923	3.2	0890	2847	6.2	0254	1572	9.2	0117	1074
0.3	9174	2752	3.3	0841	2775	6.3	0246	1548	9.3	0114	1063
0.4	8621	3448	3.4	0796	2707	6.4	0238	1525	9.4	0112	1052
0.5	0.8000	0.4000	3.5	0.0754	0.2641	6.5	0.0231	0.1503	9.5	0.0110	0.1041
0.6	7353	4411	3.6	0716	2579	6.6	0224	1481	9.6	0107	1030
0.7	6711	4697	3.7	0681	2519	6.7	0218	1460	9.7	0105	1020
0.8	6098	4878	3.8	0648	2461	6.8	0212	1439	9.8	0103	1010
0.9	5525	4972	3.9	0617	2406	6.9	0206	1419	9.9	0101	0999
1.0	0.5000	0.5000	4.0	0.0588	0.2353	7.0	0.0200	0.1400	10.0	0.0099	0.0990
1.1	4525	4977	4.1	0562	2302	7.1	0195	1381	10.5	0090	0944
1.2	4098	4918	4.2	0537	2253	7.2	0189	1363	11.0	0082	0902
1.3	3717	4832	4.3	0513	2206	7.3	0184	1435	11.5	0076	0863
1.4	3378	4729	4.4	0491	2161	7.4	0179	1327	12.0	0069	0828
1.5	0.3077	0.4615	4.5	0.0471	0,2118	7.5	0.0175	0.1310	12.5	0.0064	0.0795
1.6	2809	4494	4.6	0451	2076	7.6	0170	1293	13.0	0059	0765
1.7	2571	4370	4.7	0433	2035	7.7	0166	1277	13.5	0055	0737
1.8	2358	4245	4.8	0416	1997	7.8	0162	1261	14.0	0051	0711
1.9	2169	4121	4.9	0400	1959	7.9	0158	1246	14.5	0047	0686
2.0	0.2000	0.4000	5.0	0.0385	0.1923	8.0	0.0154	0.1231	15.0	0.0044	0.0664
2.1	1848	3881	5.1	0370	1888	8.1	0150	1216	15.5	0041	0642
2.2	1712	3767	5.2	0357	1854	8.2	0147	1202	16.0	0039	0623
2.3	1590	3657	5.3	0344	1822	8.3	0143	1187	16.5	0037	0604
2.4	1479	3550	5.4	0332	1790	8.4	0140	1174	17.0	0034	0586
2.5	0.1379	0.3448	5.5	0.0320	0.1760	8.5	0.0137	0.1160	17.5	0.0032	0.0569
2.6	1289	3351	5.6	0309	1730	8.6	0133	1147	18.0	0031	0554
2.7	1206	3257	5.7	0299	1702	8.7	0130	1134	18.5	0029	0539
2.8	1131	3167	5.8	0289	1674	8.8	0128	1122	19.0	0028	0525
2.9	1063	3082	5.9	0279	1648	8.9	0125	1110	20.0	0025	0499

Table 2.

x' positive $l' = 0$

x'	k'_t	k'_x	$C'_{\perp} \cos\varphi$ 0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	1.0000	0.8391	0.7002	0.5774	0.4663	0.3640	0.2680	0.1763	0.0875
0.2	0.9428	0.1886	0.8039	6945	5945	5016	4137	3293	2469	1654	0835
0.4	8004	3202	6286	5560	4861	4181	3511	2842	2166	1474	0757
0.6	6305	3783	4855	4374	3889	3397	2896	2378	1838	1269	0660
0.8	4762	3810	3753	3430	3091	2734	2358	1960	1533	1071	0564
1.0	0.3536	0.3536	0.2929	0.2707	0.2465	0.2203	0.1919	0.1610	0.1271	0.0897	0.0477
1.2	2623	3148	2317	2161	1985	1788	1570	1328	1057	0752	0403
1.4	1963	2748	1863	1749	1617	1466	1296	1103	0884	0634	0342
1.6	1489	2382	1520	1435	1335	1217	1081	0925	0746	0537	0292
1.8	1145	2061	1258	1194	1115	1021	0912	0784	0635	0459	0251
2.0	0.0894	0.1789	0.1056	0.1006	0.0943	0.0867	0.0777	0.0670	0.0545	0.0396	0.0218
2.2	0708	1559	0896	0857	0806	0743	0668	0578	0472	0344	0190
2.4	0569	1365	0769	0737	0695	0643	0580	0503	0412	0302	0167
2.6	0463	1203	0667	0640	0606	0561	0507	0442	0363	0266	0148
2.8	0380	1065	0583	0561	0531	0494	0447	0390	0321	0236	0132
3.0	0.0316	0.0948	0.0513	0.0495	0.0470	0.0437	0.0397	0.0347	0.0286	0.0211	0.0118
3.2	0265	0849	0455	0440	0418	0390	0354	0310	0256	0189	0106
3.4	0225	0764	0406	0393	0374	0350	0318	0279	0231	0171	0096
3.6	0192	0690	0365	0373	0337	0315	0287	0252	0209	0155	0087
3.8	0165	0612	0329	0319	0305	0286	0261	0229	0190	0141	0079
4.0	0.0143	0.0571	0.0299	0.0290	0.0277	0.0260	0.0237	0.0209	0.0173	0.0129	0.0073
4.2	0124	0522	0272	0264	0253	0237	0217	0191	0159	0118	0067
4.4	0109	0479	0249	0242	0232	0218	0199	0176	0146	0109	0061
4.6	0096	0441	0228	0222	0213	0200	0183	0162	0135	0101	0057
4.8	0085	0407	0210	0205	0196	0185	0169	0150	0125	0093	0053
5.0	0.0076	0.0377	0.0194	0.0189	0.0182	0.0171	0.0157	0.0139	0.0116	0.0086	0.0049
5.2	0067	0350	0180	0176	0169	0159	0146	0129	0108	0081	0046
5.4	0060	0326	0167	0163	0157	0148	0136	0120	0101	0075	0043
5.6	0054	0304	0156	0152	0146	0138	0127	0112	0093	0070	0040
5.8	0049	0280	0145	0142	0137	0129	0119	0105	0088	0066	0037
6.0	0.0044	0.0266	0.0136	0.0133	0.0128	0.0121	0.0111	0.0099	0.0083	0.0062	0.0035
6.2	0040	0250	0128	0125	0120	0114	0104	0093	0078	0058	0033
6.4	0037	0235	0120	0117	0113	0107	0098	0087	0073	0055	0031
6.6	0034	0222	0113	0111	0107	0101	0093	0082	0069	0052	0030
6.8	0031	0209	0106	0104	0101	0095	0088	0078	0065	0049	0028
7.0	0.0028	0.0198	0.0101	0.0099	0.0095	0.0090	0.0083	0.0074	0.0062	0.0047	0.0026
7.2	0026	0188	0095	0093	0090	0085	0078	0070	0059	0044	0025
7.4	0024	0178	0090	0088	0085	0081	0074	0066	0056	0042	0024
7.6	0022	0169	0085	0084	0081	0077	0071	0063	0053	0040	0023
7.8	0021	0160	0081	0080	0077	0073	0067	0060	0050	0038	0022
8.0	0.0019	0.0153	0.0077	0.0076	0.0073	0.0069	0.0064	0.0057	0.0048	0.0036	0.0021
8.2	0018	0146	0074	0072	0070	0066	0061	0054	0046	0034	0020
8.4	0017	0139	0070	0069	0067	0063	0058	0052	0044	0033	0019
8.6	0015	0132	0067	0066	0064	0060	0056	0050	0042	0032	0018
8.8	0014	0127	0064	0063	0061	0058	0053	0048	0040	0030	0017
9.0	0.0013	0.0121	0.0061	0.0060	0.0058	0.0055	0.0051	0.0045	0.0038	0.0029	0.0017
9.2	0013	0116	0059	0058	0056	0053	0049	0044	0037	0028	0016
9.4	0012	0111	0056	0055	0053	0051	0047	0042	0035	0027	0015
9.6	0011	0107	0054	0053	0051	0049	0045	0040	0034	0025	0015
9.8	0010	0103	0052	0051	0049	0047	0043	0039	0032	0025	0014
10.0	0.0010	0.0099	0.0050	0.0049	0.0047	0.0045	0.0041	0.0037	0.0031	0.0024	0.0014

Table 2.

x' negative $l' = 0$

$-x'$	$C'_{\perp} \cos\varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90°
0.0	1.0000	0.8391	0.7002	0.5774	0.4663	0.3640	0.2680	0.1763	0.0875	0.5000	∞
0.2	1961	9691	7850	6306	4976	3804	2746	1774	0865	4855	49.5242
0.4	3714	1.0673	8365	6534	5032	3763	2664	1689	0787	4470	12.0582
0.6	5145	1276	8532	6477	4868	3564	2476	1543	0727	3958	5.1594
0.8	6247	1540	8426	6217	4564	3276	2236	1372	0638	3424	2.7829
1.0	1.7071	1.1549	0.8138	0.5842	0.4196	0.2956	0.1986	0.1202	0.0578	0.2929	1.7072
1.2	7682	1384	7748	5420	3815	2643	1751	1047	0476	2499	1391
1.4	8137	1106	7313	4993	3450	2355	1540	0911	0408	2137	0.8067
1.6	8480	0769	6867	4584	3114	2097	1357	0796	0355	1836	5977
1.8	8741	0395	6430	4205	2812	1872	1199	0697	0309	1588	4585
2.0	1.8944	1.0007	0.6015	0.3858	0.2544	0.1675	0.1064	0.0614	0.0270	0.1382	0.3618
2.2	9104	0.9620	5625	3544	2307	1504	0948	0544	0238	1211	2921
2.4	9231	9236	5263	3261	2098	1356	0849	0484	0211	1068	2404
2.6	9333	8865	4928	3007	1914	1227	0763	0433	0188	0948	2010
2.8	9417	8506	4620	2778	1751	1115	0689	0389	0168	0846	1704
3.0	1.9487	0.8163	0.4337	0.2573	0.1607	0.1016	0.0626	0.0352	0.0151	0.0760	0.1462
3.2	9544	7837	4076	2389	1480	0930	0569	0319	0137	0685	1268
3.4	9594	7525	3837	2223	1366	0854	0521	0291	0124	0621	1109
3.6	9635	7229	3617	2072	1265	0786	0477	0266	0113	0565	0978
3.8	9671	6946	3414	1936	1173	0726	0440	0244	0104	0516	0869
4.0	1.9702	0.6681	0.3227	0.1812	0.1092	0.0673	0.0406	0.0225	0.0095	0.0473	0.0777
4.2	9728	6429	3054	1700	1018	0625	0376	0207	0088	0436	0698
4.4	9751	6188	2895	1597	0951	0582	0349	0192	0081	0402	0631
4.6	9770	5962	2747	1503	0891	0543	0324	0178	0075	0372	0573
4.8	9789	5743	2609	1417	0836	0507	0303	0166	0070	0345	0523
5.0	1.9806	0.5540	0.2482	0.1339	0.0786	0.0476	0.0283	0.0155	0.0065	0.0322	0.0478
5.2	9820	5346	2363	1266	0740	0446	0265	0145	0061	0300	0440
5.4	9833	5160	2253	1199	0698	0420	0249	0136	0057	0280	0405
5.6	9844	4985	2150	1137	0660	0396	0234	0128	0053	0263	0375
5.8	9854	4816	2054	1080	0624	0374	0221	0120	0050	0247	0348
6.0	1.9863	0.4657	0.1964	0.1027	0.0591	0.0353	0.0208	0.0113	0.0047	0.0232	0.0323
6.2	9872	4507	1880	0978	0561	0334	0197	0107	0045	0219	0302
6.4	9880	4362	1801	0932	0533	0317	0186	0101	0042	0206	0282
6.6	9887	4225	1727	0889	0507	0301	0177	0096	0040	0195	0264
6.8	9893	4092	1657	0849	0483	0286	0168	0091	0039	0185	0248
7.0	1.9899	0.3967	0.1591	0.0812	0.0461	0.0272	0.0159	0.0086	0.0036	0.0175	0.0233
7.2	9905	3848	1529	0777	0440	0260	0152	0082	0034	0166	0219
7.4	9910	3732	1471	0744	0420	0248	0144	0078	0032	0158	0207
7.6	9914	3623	1416	0713	0402	0236	0138	0074	0031	0151	0196
7.8	9918	3518	1364	0685	0385	0226	0132	0071	0030	0143	0185
8.0	1.9922	0.3417	0.1314	0.0657	0.0369	0.0216	0.0126	0.0068	0.0028	0.0137	0.0176
8.2	9926	3321	1268	0632	0354	0207	0120	0065	0027	0131	0167
8.4	9929	3228	1223	0608	0339	0199	0115	0062	0026	0125	0158
8.6	9933	3140	1182	0585	0326	0191	0111	0059	0025	0120	0151
8.8	9936	3055	1142	0563	0314	0183	0106	0057	0024	0115	0144
9.0	1.9939	0.2973	0.1104	0.0543	0.0302	0.0176	0.0102	0.0055	0.0023	0.0110	0.0137
9.2	9942	2895	1068	0524	0290	0169	0098	0052	0022	0105	0131
9.4	9945	2820	1033	0506	0280	0163	0094	0050	0021	0101	0125
9.6	9947	2747	1001	0488	0270	0157	0091	0048	0020	0097	0120
9.8	9948	2677	0970	0472	0260	0151	0087	0047	0019	0094	0115
10.0	1.9950	0.2610	0.0940	0.0456	0.0251	0.0146	0.0084	0.0045	0.0019	0.0090	0.0110

Table 2.

x' positive $l' = 1$

x'	k'_t	k'_x	$C'_\perp \cos\varphi$								
			0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.7071	0.6202	0.5351	0.4524	0.3724	0.2948	0 2193	0.1453	0.0724
0.2	0.9520	0.1904	6080	5420	4751	4080	3410	2740	2067	1390	0702
0.4	8295	3318	5146	4651	4134	3598	3047	2481	1898	1292	0662
0.6	6769	4061	4309	3940	3542	3119	2672	2201	1703	1175	0609
0.8	5308	4246	3589	3314	3009	2676	2314	1926	1506	1049	0550
1.0	0.4082	0.4082	0.2988	0.2782	0.2546	0.2282	0.1991	0.1671	0.1318	0.0927	0.0491
1.2	3125	3750	2496	2339	2156	1946	1709	1445	1148	0814	0435
1.4	2401	3361	2096	1976	1831	1662	1469	1249	0999	0713	0384
1.6	1860	2976	1773	1679	1563	1427	1267	1083	0871	0625	0338
1.8	1457	2623	1511	1436	1343	1231	1098	0943	0762	0550	0299
2.0	0.1155	0.2309	0.1297	0.1238	0.1162	0.1068	0.0957	0.0824	0.0669	0.0485	0.0265
2.2	0926	2037	1123	1074	1011	0934	0838	0725	0590	0429	0236
2.4	0751	1802	0979	0939	0886	0819	0738	0641	0523	0382	0211
2.6	0616	1601	0860	0827	0782	0724	0654	0569	0466	0341	0189
2.8	0510	1428	0759	0732	0693	0644	0583	0508	0417	0306	0170
3.0	0.0426	0.1279	0.0675	0.0652	0.0619	0.0576	0.0522	0.0456	0.0375	0.0276	0.0154
3.2	0360	1151	0604	0583	0555	0517	0470	0411	0339	0250	0139
3.4	0306	1040	0542	0525	0500	0467	0424	0372	0307	0227	0127
3.6	0262	0943	0490	0475	0453	0423	0385	0338	0280	0216	0116
3.8	0226	0858	0444	0431	0411	0385	0351	0323	0256	0189	0106
4.0	0.0196	0.0784	0.0404	0.0393	0.0375	0.0352	0.0321	0.0283	0.0234	0.0174	0.0098
4.2	0171	0719	0370	0360	0344	0323	0295	0260	0216	0160	0090
4.4	0150	0661	0339	0330	0316	0297	0271	0239	0199	0148	0084
4.6	0133	0610	0312	0304	0292	0274	0251	0221	0184	0137	0077
4.8	0118	0564	0288	0281	0269	0253	0232	0205	0171	0127	0072
5.0	0.0105	0.0523	0.0267	0.0260	0.0250	0.0235	0.0216	0.0191	0.0159	0.0119	0.0067
5.2	0094	0487	0248	0242	0232	0219	0201	0178	0148	0111	0062
5.4	0084	0453	0231	0225	0216	0204	0187	0166	0139	0104	0059
5.6	0076	0424	0215	0210	0202	0190	0175	0155	0130	0097	0055
5.8	0068	0397	0201	0197	0189	0179	0164	0145	0122	0091	0052
6.0	0.0062	0.0372	0.0189	0.0184	0.0178	0.0167	0.0154	0.0137	0.0114	0.0086	0.0049
6.2	0056	0350	0177	0173	0167	0157	0145	0129	0108	0081	0046
6.4	0051	0329	0167	0163	0157	0148	0137	0121	0102	0076	0043
6.6	0047	0310	0157	0154	0148	0140	0129	0115	0096	0072	0041
6.8	0043	0293	0148	0145	0140	0132	0122	0108	0091	0068	0039
7.0	0.0040	0.0277	0.0140	0.0137	0.0132	0.0125	0.0115	0.0103	0.0086	0.0065	0.0037
7.2	0036	0263	0133	0130	0125	0119	0109	0097	0082	0061	0035
7.4	0034	0249	0126	0124	0119	0113	0104	0092	0078	0058	0033
7.6	0031	0237	0119	0117	0113	0107	0099	0088	0074	0055	0032
7.8	0029	0225	0113	0111	0107	0102	0094	0084	0070	0053	0030
8.0	0.0027	0.0214	0.0108	0.0106	0.0102	0.0097	0.0089	0.0080	0.0067	0.0050	0.0029
8.2	0025	0204	0103	0101	0098	0092	0085	0076	0064	0048	0028
8.4	0023	0195	0098	0096	0093	0088	0082	0073	0061	0046	0026
8.6	0022	0186	0094	0092	0089	0084	0078	0069	0058	0044	0025
8.8	0020	0178	0090	0088	0085	0081	0075	0066	0056	0042	0024
9.0	0.0019	0.0170	0.0086	0.0084	0.0081	0.0077	0.0071	0.0064	0.0054	0.0041	0.0023
9.2	0018	0163	0082	0081	0078	0074	0069	0061	0051	0039	0022
9.4	0017	0157	0079	0077	0075	0071	0066	0059	0049	0037	0021
9.6	0016	0150	0075	0074	0072	0068	0063	0056	0047	0036	0020
9.8	0015	0144	0073	0071	0069	0065	0061	0054	0046	0034	0020
10.0	0.0014	0.0139	0.0070	0.0068	0.0066	0.0063	0.0058	0.0052	0.0044	0.0033	0.0019

Table 2.

x' negative $l' = 1$

$-x'$	$C'_{\perp}$ cos φ									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	−90°
0.0	0.7071	0.6202	0.5351	0.4524	0.3724	0.2948	0.2193	0.1453	0.0724	0.4142	2.4143
0.2	8061	6941	5879	4885	3955	3081	2257	1472	0723	4078	3118
0.4	8995	7584	6298	5134	4081	3126	2253	1448	0701	3896	0486
0.6	9833	8103	6581	5257	4101	3088	2189	1386	0662	3630	1.7169
0.8	1.0553	8482	6731	5264	4029	2981	2081	1299	0612	3316	3931
1.0	1.1153	0.8734	0.6764	0.5176	0.3887	0.2829	0.1946	0.1199	0.0558	0.2988	1.1152
1.2	1646	8871	6702	5020	3701	2650	1798	1094	0503	2671	0.8921
1.4	2045	8917	6570	4818	3489	2461	1648	0991	0451	2377	7178
1.6	2369	8894	6390	4589	3268	2273	1504	0896	0405	2112	5833
1.8	2632	8816	6178	4349	3048	2092	1370	0809	0362	1878	4792
2.0	1,2845	0.8701	0.5948	0.4107	0.2835	0.1923	0.1246	0.0730	0.0325	0.1674	0.3983
2.2	3020	8557	5709	3870	2634	1767	1136	0660	0292	1496	3348
2.4	3163	8391	5468	3642	2446	1625	1035	0598	0263	1341	2843
2.6	3284	8215	5230	3425	2273	1495	0946	0543	0237	1207	2438
2.8	3383	8027	4997	3221	2113	1378	0866	0495	0215	1090	2110
3.0	1.3468	0.7837	0.4773	0.3030	0.1966	0.1272	0.0795	0.0452	0.0196	0.0988	0.1841
3.2	3538	7643	4557	2852	1832	1177	0731	0414	0179	0899	1618
3.4	3600	7449	4351	2687	1710	1091	0674	0380	0164	0820	1432
3.6	3653	7258	4155	2533	1598	1013	0623	0350	0150	0751	1275
3.8	3699	7068	3969	2390	1496	0943	0577	0323	0138	0690	1142
4.0	1.3738	0.6886	0.3793	0.2258	0.1402	0.0879	0.0536	0.0299	0.0128	0.0636	0.1028
4.2	3772	6699	3626	2135	1316	0821	0499	0277	0118	0587	0930
4.4	3802	6520	3469	2021	1238	0768	0465	0258	0110	0544	0845
4.6	3830	6346	3319	1915	1165	0721	0435	0240	0102	0506	0771
4.8	3853	6177	3179	1817	1099	0677	0407	0225	0095	0471	0706
5.0	1.3875	0.6012	0.3045	0.1725	0.1038	0.0637	0.0382	0.0210	0.0089	0.0439	0.0649
5.2	3893	5854	2920	1640	0981	0600	0359	0197	0083	0411	0598
5.4	3911	5699	2802	1561	0929	0566	0338	0185	0078	0385	0553
5.6	3927	5550	2689	1487	0881	0535	0319	0175	0073	0361	0513
5.8	3941	5403	2583	1418	0836	0506	0301	0164	0069	0340	0477
6.0	1.3954	0.5264	0.2483	0.1353	0.0795	0.0480	0.0285	0.0155	0.0065	0.0320	0.0444
6.2	3965	5129	2388	1293	0756	0455	0270	0147	0062	0302	0415
6.4	3975	4996	2298	1236	0720	0433	0256	0139	0058	0286	0388
6.6	3985	4870	2213	1183	0687	0411	0243	0132	0055	0270	0364
6.8	3994	4746	2132	1133	0656	0392	0231	0125	0052	0256	0342
7.0	1.4002	0.4628	0.2055	0.1086	0.0626	0.0373	0.0220	0.0119	0.0050	0.0243	0.0322
7.2	4009	4513	1982	1042	0599	0356	0209	0113	0047	0231	0304
7.4	4016	4401	1914	1000	0573	0341	0200	0108	0045	0220	0287
7.6	4023	4294	1847	0961	0549	0326	0191	0103	0043	0210	0272
7.8	4029	4189	1784	0924	0527	0312	0182	0098	0041	0200	0258
8.0	1.4034	0.4089	0.1725	0.0889	0.0505	0.0299	0.0174	0.0094	0.0039	0.0191	0.0244
8.2	4039	3992	1668	0856	0485	0286	0167	0090	0037	0182	0232
8.4	4044	3897	1614	0825	0467	0275	0160	0086	0036	0174	0221
8.6	4049	3806	1562	0795	0449	0264	0154	0083	0034	0167	0210
8.8	4053	3718	1513	0767	0432	0253	0148	0079	0033	0160	0201
9.0	1.4057	0.3633	0.1466	0.0740	0.0416	0.0244	0.0142	0.0076	0.0032	0.0154	0.0191
9.2	4060	3550	1421	0715	0401	0235	0136	0073	0030	0147	0183
9.4	4063	3469	1378	0691	0387	0226	0131	0070	0029	0142	0175
9.6	4067	3392	1337	0668	0373	0218	0126	0068	0028	0136	0167
9.8	4070	3317	1297	0646	0360	0210	0122	0065	0027	0131	0160
10.0	1.4073	0.3244	0.1260	0.0625	0.0348	0.0203	0.0117	0.0063	0.0026	0.0126	0.0154

Table 2.

x′ positive $l' = 2$

x'	k'_t	k'_x	$C'_\perp \cos\varphi$ 0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.4472	0.4087	0.3645	0.3165	0.2661	0.2141	0.1612	0.1077	0.0539
0.2	0.9577	0.1915	4074	3750	3373	2956	2509	2039	1551	1047	0530
0.4	8486	3394	3684	3415	3094	2735	2341	1920	1475	1006	0514
0.6	7102	4260	3313	3089	2817	2508	2164	1791	1387	0954	0493
0.8	5742	4593	2966	2779	2551	2285	1987	1656	1293	0897	0467
1.0	0.4564	0.4564	0.2646	0.2492	0.2299	0.2072	0.1812	0.1521	0.1196	0.0837	0.0439
1.2	3611	4333	2357	2229	2066	1872	1647	1390	1101	0775	0410
1.4	2863	4008	2099	1992	1854	1688	1492	1265	1008	0715	0381
1.6	2284	3655	1870	1780	1663	1519	1349	1151	0921	0656	0352
1.8	1837	3307	1668	1593	1493	1369	1220	1045	0840	0602	0325
2.0	0.1491	0.2981	0.1491	0.1427	0.1342	0.1234	0.1103	0.0948	0.0766	0.0552	0.0299
2.2	1220	2685	1336	1282	1208	1114	0999	0862	0699	0505	0275
2.4	1008	2420	1200	1154	1090	1008	0907	0784	0637	0463	0253
2.6	0840	2185	1082	1042	0986	0914	0824	0715	0583	0424	0233
2.8	0706	1976	0978	0944	0894	0831	0750	0652	0534	0389	0215
3.0	0.0598	0.1793	0.0886	0.0857	0.0813	0.0757	0.0685	0.0597	0.0489	0.0358	0.0198
3.2	0510	1631	0806	0780	0742	0691	0627	0548	0450	0330	0183
3.4	0437	1487	0736	0713	0679	0634	0575	0503	0414	0304	0169
3.6	0378	1361	0673	0653	0623	0581	0529	0463	0382	0282	0157
3.8	0329	1248	0618	0600	0573	0536	0488	0428	0353	0261	0146
4.0	0.0287	0.1148	0.0568	0.0553	0.0528	0.0494	0.0451	0.0396	0.0327	0.0242	0.0135
4.2	0252	1059	0524	0510	0488	0458	0418	0367	0304	0225	0126
4.4	0222	0979	0485	0473	0452	0425	0388	0342	0283	0210	0118
4.6	0197	0908	0450	0438	0420	0394	0361	0318	0264	0196	0110
4.8	0176	0843	0418	0408	0391	0367	0337	0297	0247	0183	0103
5.0	0.0157	0.0785	0.0389	0.0380	0.0365	0.0343	0.0314	0.0278	0.0231	0.0172	0.0097
5.2	0141	0732	0364	0355	0341	0321	0294	0260	0216	0161	0091
5.4	0127	0685	0340	0332	0319	0301	0276	0244	0203	0152	0086
5.6	0115	0642	0319	0311	0300	0282	0259	0229	0191	0143	0080
5.8	0104	0602	0299	0293	0282	0265	0244	0216	0180	0135	0076
6.0	0.0094	0.0566	0.0281	0.0276	0.0265	0.0250	0.0230	0.0204	0.0170	0.0127	0.0072
6.2	0086	0533	0265	0260	0250	0236	0217	0192	0161	0120	0068
6.4	0078	0503	0250	0245	0236	0223	0205	0182	0152	0114	0065
6.6	0072	0475	0236	0232	0223	0211	0194	0172	0144	0108	0061
6.8	0066	0450	0224	0219	0211	0200	0184	0163	0137	0102	0058
7.0	0.0061	0.0426	0.0212	0.0208	0.0201	0.0190	0.0175	0.0155	0.0130	0.0097	0.0055
7.2	0056	0404	0201	0197	0190	0180	0166	0147	0123	0093	0053
7.4	0052	0384	0191	0187	0181	0171	0158	0140	0118	0088	0050
7.6	0048	0365	0182	0178	0172	0163	0150	0134	0112	0084	0048
7.8	0045	0347	0173	0170	0164	0155	0143	0128	0107	0080	0046
8.0	0.0041	0.0331	0.0165	0.0162	0.0156	0.0148	0.0137	0.0122	0.0102	0.0077	0.0044
8.2	0039	0316	0158	0155	0149	0142	0131	0116	0098	0073	0042
8.4	0036	0302	0150	0148	0143	0135	0125	0111	0093	0070	0040
8.6	0033	0289	0144	0141	0137	0129	0120	0106	0090	0067	0038
8.8	0032	0276	0138	0135	0131	0124	0115	0102	0086	0065	0037
9.0	0.0029	0.0265	0.0132	0.0130	0.0125	0.0119	0.0110	0.0098	0.0082	0.0062	0.0035
9.2	0028	0254	0127	0124	0120	0114	0105	0094	0079	0060	0034
9.4	0026	0243	0121	0119	0115	0109	0101	0090	0076	0057	0033
9.6	0024	0234	0117	0115	0111	0105	0097	0087	0073	0055	0032
9.8	0023	0225	0112	0110	0107	0101	0094	0084	0070	0053	0030
10.0	0.0022	0.0216	0.0108	0.0106	0.0103	0.0097	0.0090	0.0081	0.0068	0.0051	0.0029

Table 2.

x′ negative $l' = 2$

$-x'$	$C'_{\perp} \cos \varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90°
0.0	0.4472	0.4087	0.3645	0.3165	0.2661	0.2141	0.1612	0.1077	0.0539	0.3090	0.8090
0.2	4871	4415	3901	3354	2790	2221	1654	1094	0542	3069	8000
0.4	5259	4726	4133	3515	2891	2276	1675	1095	0536	3009	7741
0.6	5632	5012	4336	3644	2961	2303	1676	1083	0524	2913	7344
0.8	5978	5266	4502	3737	2999	2304	1657	1059	0507	2790	6848
1.0	0.6298	0.5487	0.4632	0.3795	0.3006	0.2281	0.1621	0.1024	0.0487	0.2646	0.6298
1.2	6587	5674	4726	3818	2985	2236	1571	0982	0462	2491	5730
1.4	6845	5825	4785	3812	2940	2176	1511	0936	0436	2330	5174
1.6	7075	5944	4812	3778	2876	2101	1444	0886	0408	2169	4649
1.8	7276	6033	4813	3724	2796	2019	1373	0834	0382	2013	4165
2.0	0.7454	0.6096	0.4789	0.3652	0.2705	0.1931	0.1301	0.0783	0.0356	0.1863	0.3727
2.2	7608	6137	4745	3565	2608	1841	1227	0734	0331	1723	3336
2.4	7745	6157	4687	3469	2505	1751	1157	0686	0307	1593	2990
2.6	7862	6160	4613	3366	2401	1661	1088	0641	0286	1472	2684
2.8	7966	6149	4531	3259	2297	1573	1023	0598	0265	1362	2416
3.0	0.8057	0.6125	0.4441	0.3149	0.2194	0.1489	0.0962	0.0559	0.0247	0.1260	0.2180
3.2	8139	6092	4345	3039	2094	1409	0903	0522	0230	1168	1973
3.4	8209	6048	4245	2929	1997	1333	0849	0488	0214	1084	1790
3.6	8271	5999	4142	2820	1904	1261	0798	0457	0199	1007	1630
3.8	8326	5942	4037	2714	1815	1193	0752	0429	0186	0938	1488
4.0	0.8375	0.5882	0.3933	0.2611	0.1730	0.1129	0.0708	0.0402	0.0174	0.0874	0.1362
4.2	8419	5817	3828	2511	1649	1070	0667	0377	0163	0816	1250
4.4	8459	5749	3725	2414	1572	1013	0629	0355	0153	0763	1151
4.6	8494	5679	3622	2321	1499	0961	0594	0334	0143	0715	1062
4.8	8526	5605	3521	2232	1431	0912	0561	0315	0135	0671	0983
5.0	0.8555	0.5532	0.3423	0.2147	0.1366	0.0866	0.0531	0.0297	0.0127	0.0630	0.0912
5.2	8580	5456	3326	2064	1304	0823	0503	0280	0119	0593	0848
5.4	8603	5379	3232	1986	1246	0783	0477	0265	0113	0559	0790
5.6	8626	5303	3141	1912	1192	0746	0453	0251	0106	0528	0737
5.8	8645	5226	3051	1840	1141	0710	0430	0238	0101	0498	0690
6.0	0.8663	0.5150	0.2965	0.1772	0.1092	0.0678	0.0409	0.0226	0.0097	0.0472	0.0646
6.2	8679	5072	2880	1707	1046	0645	0389	0214	0091	0447	0607
6.4	8693	4995	2799	1645	1003	0618	0371	0204	0086	0424	0571
6.6	8708	4919	2720	1585	0962	0591	0354	0194	0082	0403	0538
6.8	8721	4843	2644	1529	0923	0565	0338	0185	0078	0383	0507
7.0	0.8731	0.4768	0.2570	0.1475	0.0887	0.0541	0.0323	0.0176	0.0074	0.0364	0.0479
7.2	8744	4694	2499	1424	0852	0518	0309	0168	0071	0347	0454
7.4	8752	4620	2429	1375	0819	0497	0295	0161	0068	0331	0430
7.6	8763	4548	2363	1328	0788	0477	0283	0154	0065	0316	0408
7.8	8771	4475	2299	1284	0759	0458	0271	0148	0062	0302	0387
8.0	0.8780	0.4405	0.2236	0.1241	0.0731	0.0440	0.0260	0.0141	0.0059	0.0289	0.0368
8.2	8787	4335	2177	1200	0705	0423	0250	0135	0057	0277	0351
8.4	8795	4266	2118	1162	0679	0408	0240	0130	0054	0265	0334
8.6	8802	4199	2062	1125	0656	0392	0231	0125	0052	0254	0319
8.8	8808	4132	2008	1089	0633	0378	0222	0120	0050	0244	0305
9.0	0.8813	0.4066	0.1956	0.1055	0.0611	0.0364	0.0214	0.0115	0.0048	0.0235	0.0291
9.2	8819	4002	1906	1023	0591	0351	0206	0111	0046	0226	0279
9.4	8823	3938	1857	0991	0571	0339	0199	0107	0045	0217	0267
9.6	8827	3875	1810	0962	0552	0328	0191	0103	0043	0209	0256
9.8	8831	3813	1764	0933	0535	0317	0185	0100	0041	0201	0246
10.0	0.8835	0.3753	0.1720	0.0906	0.0518	0.0306	0.0179	0.0096	0.0040	0.0194	0.0236

Table 2.

x′ positive $l' = 3$

x'	k'_t	k'_x	$C'_\perp \cos\varphi$ 0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.3162	0.2952	0.2682	0.2365	0.2013	0.1636	0.1241	0.0834	0.0419
0.2	0.9596	0.1919	2963	2777	2536	2250	1928	1579	1207	0817	0413
0.4	8553	3421	2765	2602	2387	2130	1837	1515	1167	0796	0406
0.6	7224	4334	2573	2429	2238	2007	1742	1445	1120	0770	0396
0.8	5912	4729	2387	2261	2091	1885	1644	1372	1070	0741	0384
1.0	0.4767	0.4767	0.2209	0.2099	0.1949	0.1763	0.1546	0.1297	0.1010	0.0709	0.0370
1.2	3831	4598	2040	1944	1811	1646	1448	1222	0964	0676	0355
1.4	3089	4324	1882	1797	1680	1532	1354	1147	0910	0641	0339
1.6	2506	4010	1735	1660	1556	1423	1263	1074	0857	0607	0322
1.8	2049	3689	1598	1532	1440	1321	1176	1005	0804	0572	0306
2.0	0.1690	0.3381	0.1472	0.1414	0.1332	0.1225	0.1094	0.0938	0.0754	0.0538	0.0289
2.2	1405	3092	1356	1305	1232	1136	1017	0875	0705	0506	0273
2.4	1178	2828	1251	1205	1140	1054	0946	0816	0660	0476	0257
2.6	0996	2588	1154	1114	1055	0977	0880	0760	0617	0446	0243
2.8	0847	2371	1066	1030	0978	0907	0818	0709	0576	0419	0228
3.0	0.0725	0.2176	0.0986	0.0954	0.0907	0.0842	0.0761	0.0661	0.0540	0.0392	0.0215
3.2	0626	2001	0913	0885	0841	0784	0709	0617	0505	0368	0202
3.4	0542	1844	0847	0821	0783	0729	0661	0576	0473	0345	0190
3.6	0473	1702	0787	0764	0728	0680	0617	0539	0442	0324	0179
3.8	0414	1574	0732	0711	0679	0634	0576	0504	0415	0304	0168
4.0	0.0365	0.1459	0.0682	0.0663	0.0633	0.0593	0.0540	0.0473	0.0389	0.0287	0.0159
4.2	0323	1355	0636	0619	0592	0555	0505	0443	0366	0269	0150
4.4	0287	1261	0594	0579	0554	0519	0474	0416	0344	0254	0141
4.6	0255	1176	0556	0542	0519	0487	0445	0391	0324	0240	0134
4.8	0229	1099	0522	0509	0487	0458	0419	0369	0305	0226	0127
5.0	0.0206	0.1028	0.0490	0.0478	0.0459	0.0431	0.0394	0.0347	0.0288	0.0213	0.0120
5.2	0185	0963	0460	0449	0432	0406	0372	0328	0272	0202	0113
5.4	0168	0905	0434	0423	0407	0383	0351	0309	0257	0191	0107
5.6	0152	0851	0409	0400	0384	0361	0331	0293	0243	0181	0102
5.8	0138	0801	0386	0377	0363	0342	0314	0277	0231	0172	0097
6.0	0.0126	0.0756	0.0365	0.0357	0.0343	0.0324	0.0297	0.0263	0.0219	0.0163	0.0092
6.2	0115	0725	0345	0338	0325	0307	0282	0249	0208	0155	0087
6.4	0105	0676	0327	0320	0308	0291	0268	0237	0198	0148	0083
6.6	0097	0640	0310	0304	0293	0277	0254	0225	0188	0140	0080
6.8	0089	0607	0295	0289	0278	0263	0242	0215	0179	0134	0076
7.0	0.0082	0.0576	0.0280	0.0275	0.0265	0.0250	0.0231	0.0204	0.0171	0.0128	0.0072
7.2	0076	0548	0267	0262	0253	0238	0220	0195	0163	0122	0069
7.4	0070	0521	0254	0250	0240	0228	0209	0186	0156	0117	0066
7.6	0065	0497	0243	0238	0230	0217	0200	0178	0149	0112	0063
7.8	0061	0474	0232	0227	0219	0208	0191	0170	0142	0107	0060
8.0	0.0057	0.0453	0.0221	0.0217	0.0210	0.0199	0.0183	0.0163	0.0136	0.0103	0.0058
8.2	0053	0433	0212	0208	0201	0190	0175	0156	0131	0098	0056
8.4	0049	0414	0203	0199	0192	0182	0168	0150	0125	0094	0054
8.6	0046	0396	0194	0191	0184	0175	0161	0144	0120	0090	0051
8.8	0043	0379	0186	0183	0177	0168	0155	0138	0116	0087	0049
9.0	0.0040	0.0364	0.0179	0.0176	0.0170	0.0161	0.0149	0.0133	0.0111	0.0084	0.0048
9.2	0038	0349	0172	0169	0163	0155	0143	0127	0107	0080	0046
9.4	0036	0335	0165	0162	0157	0149	0137	0122	0103	0077	0044
9.6	0033	0322	0159	0156	0151	0143	0132	0118	0099	0075	0043
9.8	0032	0310	0153	0150	0146	0138	0127	0114	0096	0072	0041
10.0	0.0030	0.0298	0.0147	0.0145	0.0140	0.0133	0.0123	0.0110	0.0092	0.0069	0.0039

Table 2.

x' negative $l' = 3$

$-x'$	$C'_\perp \cos\varphi$									$C'_\perp$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90'
0.0	0.3162	0.2952	0.2682	0.2365	0.2013	0.1636	0.1241	0.0834	0.0419	0.2403	0.4625
0.2	3362	3125	2822	2473	2090	1686	1268	0845	0421	2394	4602
0.4	3559	3293	2956	2571	2157	1725	1288	0850	0420	2369	4535
0.6	3752	3453	3081	2660	2213	1755	1304	0850	0417	2329	4428
0.8	3938	3605	3194	2736	2256	1773	1300	0844	0410	2275	4286
1.0	0.4116	0.3748	0.3295	0.2798	0.2287	0.1782	0.1295	0.0833	0.0401	0.2209	0.4116
1.2	4284	3878	3384	2848	2306	1779	1282	0817	0391	2133	3925
1.4	4442	3997	3458	2884	2313	1768	1262	0798	0379	2051	3720
1.6	4590	4103	3519	2907	2308	1748	1236	0775	0364	1964	3507
1.8	4727	4196	3568	2917	2293	1720	1205	0750	0350	1874	3294
2.0	0.4853	0.4277	0.3603	0.2917	0.2270	0.1686	0.1171	0.0723	0.0336	0.1782	0.3083
2.2	4968	4348	3627	2905	2238	1647	1134	0695	0320	1692	2878
2.4	5074	4407	3639	2884	2199	1603	1095	0667	0305	1603	2682
2.6	5171	4456	3643	2856	2156	1557	1055	0637	0290	1516	2497
2.8	5259	4496	3636	2820	2108	1509	1014	0609	0276	1433	2322
3.0	0.5339	0.4527	0.3623	0.2779	0.2055	0.1459	0.0974	0.0581	0.0261	0.1354	0.2160
3.2	5173	4551	3602	2733	2001	1409	0933	0553	0247	1278	2009
3.4	5478	4567	3574	2683	1946	1358	0894	0527	0235	1207	1869
3.6	5538	4577	3542	2629	1889	1308	0855	0502	0222	1139	1741
3.8	5593	4581	3505	2573	1831	1268	0818	0477	0211	1076	1622
4.0	0.5643	0.4581	0.3464	0.2516	0.1774	0.1210	0.0782	0.0454	0.0200	0.1017	0.1513
4.2	5688	4575	3420	2457	1717	1163	0748	0432	0189	0961	1413
4.4	5730	4565	3374	2397	1662	1118	0714	0411	0180	0909	1321
4.6	5768	4552	3324	2338	1607	1074	0683	0391	0171	0861	1236
4.8	5804	4535	3273	2279	1553	1032	0653	0373	0162	0815	1159
5.0	0.5835	0.4516	0.3220	0.2219	0.1500	0.0990	0.0624	0.0355	0.0154	0.0773	0.1087
5.2	5865	4494	3167	2160	1450	0952	0597	0339	0147	0733	1022
5.4	5891	4469	3113	2103	1400	0914	0571	0323	0139	0696	0961
5.6	5916	4443	3058	2046	1353	0879	0547	0308	0132	0662	0905
5.8	5938	4414	3003	1990	1307	0844	0524	0294	0126	0629	0854
6.0	0.5960	0.4385	0.2948	0.1935	0.1263	0.0812	0.0502	0.0281	0.0120	0.0599	0.0806
6.2	5979	4354	2893	1882	1220	0781	0481	0269	0115	0571	0762
6.4	5998	4321	2838	1830	1179	0751	0461	0257	0110	0544	0722
6.6	6014	4288	2784	1780	1139	0723	0442	0246	0105	0519	0684
6.8	6030	4253	2730	1731	1101	0696	0425	0236	0100	0496	0649
7.0	0.6045	0.4219	0.2677	0.1683	0.1065	0.0670	0.0408	0.0226	0.0096	0.0474	0.0616
7.2	6058	4183	2624	1636	1030	0646	0392	0217	0092	0454	0586
7.4	6070	4146	2572	1591	0996	0622	0377	0208	0088	0434	0558
7.6	6082	4109	2521	1547	0964	0600	0363	0200	0085	0416	0531
7.8	6092	4071	2470	1505	0933	0579	0349	0192	0081	0399	0507
8.0	0.6103	0.4034	0.2421	0.1464	0.0904	0.0559	0.0336	0.0185	0.0078	0.0383	0.0484
8.2	6113	3996	2373	1425	0875	0540	0324	0178	0075	0368	0462
8.4	6122	3957	2325	1386	0848	0521	0312	0171	0072	0353	0442
8.6	6130	3919	2278	1349	0822	0504	0301	0165	0069	0340	0423
8.8	6139	3880	2232	1313	0796	0487	0291	0159	0067	0327	0405
9.0	0.6146	0.3842	0.2187	0.1279	0.0772	0.0471	0.0281	0.0153	0.0064	0.0315	0.0388
9.2	6153	3803	2143	1245	0749	0456	0271	0148	0062	0303	0372
9.4	6159	3764	2100	1213	0727	0442	0262	0143	0060	0292	0358
9.6	6166	3726	2058	1181	0706	0428	0253	0138	0058	0282	0344
9.8	6171	3687	2017	1151	0685	0414	0245	0133	0056	0272	0330
10.0	0.6177	0.3649	0.1977	0.1122	0.0665	0.0402	0.0237	0.0129	0.0054	0.0262	0.0318

Table 2.

x' positive $l' = 4$

x'	k'_t	k'_x	$C'_\perp \cos\varphi$								
			0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.2425	0.2292	0.2105	0.1873	0.1607	0.1315	0.1002	0.0676	0.0340
0.2	0.9604	0.1921	2308	2186	2015	1802	1554	1278	0981	0665	0337
0.4	8581	3432	2191	2081	1924	1728	1497	1238	0956	0652	0332
0.6	7276	4365	2076	1976	1834	1653	1439	1196	0928	0637	0327
0.8	5986	4789	1963	1873	1742	1577	1378	1152	0898	0620	0320
1.0	0.4859	0.4859	0.1853	0.1772	0.1653	0.1501	0.1317	0.1105	0.0866	0.0602	0.0313
1.2	3935	4722	1747	1674	1565	1426	1256	1058	0833	0582	0303
1.4	3199	4478	1646	1578	1480	1352	1195	1011	0800	0561	0294
1.6	2619	4189	1548	1488	1397	1280	1135	0964	0765	0540	0284
1.8	2161	3890	1455	1400	1319	1211	1077	0917	0731	0518	0275
2.0	0.1799	0.3599	0.1367	0.1317	0.1243	0.1144	0.1020	0.0872	0.0697	0.0496	0.0265
2.2	1510	3324	1284	1239	1171	1079	0965	0828	0664	0474	0253
2.4	1278	3068	1206	1165	1102	1018	0913	0785	0632	0453	0243
2.6	1090	2835	1132	1095	1038	0961	0863	0744	0601	0431	0233
2.8	0936	2620	1063	1029	0977	0906	0816	0704	0571	0411	0223
3.0	0.0809	0.2426	0.0999	0.0968	0.920	0.0855	0.0771	0.0667	0.0542	0.0391	0.0213
3.2	0703	2249	0939	0911	0867	0806	0728	0631	0514	0372	0203
3.4	0614	2088	0883	0857	0816	0760	0688	0598	0488	0354	0194
3.6	0539	1943	0830	0807	0770	0718	0650	0566	0463	0337	0185
3.8	0477	1810	0782	0760	0726	0678	0615	0536	0439	0321	0176
4.0	0.0422	0.1689	0.0736	0.0717	0.0685	0.640	0.0581	0.0508	0.0417	0.0305	0.0168
4.2	0376	1578	0695	0677	0647	0605	0551	0481	0396	0290	0160
4.4	0336	1478	0656	0639	0611	0573	0522	0457	0376	0276	0152
4.6	0301	1386	0619	0604	0579	0542	0495	0433	0357	0262	0146
4.8	0271	1301	0586	0571	0547	0513	0469	0411	0339	0250	0139
5.0	0.0245	0.1223	0.0554	0.0541	0.0519	0.0487	0.0445	0.0391	0.0323	0.0239	0.0132
5.2	0222	1152	0525	0513	0492	0462	0422	0372	0308	0227	0127
5.4	0201	1086	0498	0487	0467	0439	0402	0354	0293	0217	0121
5.6	0183	1026	0473	0462	0443	0417	0382	0337	0279	0207	0116
5.8	0167	0970	0448	0439	0422	0397	0364	0321	0266	0198	0111
6.0	0.0153	0.0918	0.0426	0.0417	0.0402	0.0378	0.0347	0.0306	0.0254	0.0189	0.0106
6.2	1041	0870	0406	0398	0382	0360	0331	0292	0243	0180	0101
6.4	0129	0826	0387	0379	0364	0343	0315	0279	0232	0172	0097
6.6	0119	0785	0368	0361	0348	0328	0301	0267	0221	0165	0093
6.8	0110	0746	0352	0344	0332	0313	0288	0255	0212	0158	0089
7.0	0.0102	0.0711	0.0335	0.0329	0.0317	0.0299	0.0275	0.0244	0.0203	0.0152	0.0085
7.2	0094	0677	0321	0315	0303	0286	0263	0233	0195	0145	0082
7.4	0087	0646	0307	0301	0290	0274	0252	0223	0187	0139	0078
7.6	0081	0617	0294	0288	0278	0263	0242	0214	0179	0134	0075
7.8	0076	0589	0281	0276	0266	0252	0232	0206	0172	0129	0072
8.0	0.0071	0.0564	0.0269	0.0265	0.0255	0.0241	0.0223	0.0197	0.0165	0.0124	0.0070
8.2	0066	0540	0258	0254	0245	0232	0214	0190	0159	0119	0067
8.4	0062	0517	0248	0244	0235	0223	0205	0182	0153	0115	0065
8.6	0057	0496	0238	0234	0226	0214	0198	0176	0147	0110	0062
8.8	0054	0476	0229	0225	0218	0206	0190	0169	0142	0106	0060
9.0	0.0051	0.0457	0.0220	0.0217	0.0209	0.0198	0.0183	0.0163	0.0137	0.0102	0.0058
9.2	0048	0439	0212	0209	0202	0191	0176	0157	0132	0099	0056
9.4	0045	0423	0204	0201	0194	0184	0170	0151	0127	0095	0054
9.6	0042	0406	0197	0194	0187	0177	0164	0146	0122	0092	0052
9.8	0040	0392	0190	0187	0180	0171	0158	0141	0118	0089	0050
10.0	0.0038	0.0377	0.0183	0.0180	0.0174	0.0165	0.0153	0.0136	0.0114	0.0086	0.0049

Table 2.

x' negative $l' = 4$

$-x'$	$C'_{\perp} \cos\varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	−90°
0.0	0,2425	0.2292	0.2105	0.1873	0.1607	0.1315	0.1002	0.0676	0.0340	0.1952	0.3202
0.2	2543	2396	2191	1942	1654	1347	1021	0683	0342	1948	3193
0.4	2659	2499	2276	2007	1702	1376	1035	0687	0341	1935	3167
0.6	2775	2600	2357	2066	1743	1399	1046	0691	0340	1915	3125
0.8	2887	2696	2434	2121	1778	1418	1052	0690	0338	1888	3068
1.0	0.2997	0.2789	0.2505	0.2171	0.1808	0.1431	0.1054	0.0687	0.0334	0.1854	0.2997
1.2	3103	2878	2571	2214	1831	1439	1052	0681	0328	1814	2915
1.4	3205	2961	2632	2252	1848	1442	1047	0672	0323	1769	2823
1.6	3303	3039	2686	2282	1860	1440	1039	0662	0316	1719	2724
1.8	3396	3112	2735	2308	1866	1434	1026	0650	0308	1667	2619
2.0	0.3484	0.3180	0.2777	0.2326	0.1866	0.1423	0.1008	0.0636	0.0300	0.1612	0.2511
2.2	3567	3241	2813	2338	1862	1409	0993	0621	0290	1555	2402
2.4	3645	3298	2843	2345	1852	1391	0974	0604	0281	1498	2292
2.6	3719	3349	2866	2346	1838	1369	0952	0587	0272	1440	2183
2.8	3788	3394	2885	2342	1820	1346	0930	0570	0262	1382	2076
3.0	0.3852	0.3435	0.2898	0.2334	0.1798	0.1320	0.0906	0.0551	0.0252	0.1326	0.1973
3.2	3912	3471	2906	2321	1774	1292	0881	0533	0243	1270	1872
3.4	3969	3502	2910	2305	1747	1264	0856	0515	0233	1216	1776
3.6	4020	3529	2910	2284	1718	1233	0829	0497	0224	1164	1684
3.8	4069	3552	2905	2261	1687	1202	0804	0479	0215	1113	1596
4.0	0.4114	0.3571	0.2898	0.2236	0.1654	0.1171	0.0778	0.0462	0.0207	0.1064	0.1513
4.2	4156	3587	2886	2207	1621	1140	0753	0445	0198	1017	1434
4.4	4195	3600	2873	2178	1586	1108	0728	0428	0190	0973	1359
4.6	4232	3610	2855	2146	1551	1076	0703	0412	0182	0930	1289
4.8	4265	3616	2836	2113	1516	1045	0680	0396	0175	0889	1223
5.0	0.4297	0.3621	0.2815	0.2078	0.1480	0.1014	0.0657	0.0381	0.0167	0.0850	0.1161
5.2	4325	3623	2791	2043	1444	0984	0634	0367	0160	0814	1102
5.4	4353	3622	2766	2008	1409	0954	0612	0353	0154	0779	1047
5.6	4379	3620	2739	1971	1373	0925	0591	0339	0148	0745	0996
5.8	4402	3616	2712	1935	1338	0896	0570	0326	0142	0714	0947
6.0	0.4424	0.3610	0.2682	0.1898	0.1304	0.0869	0.0550	0.0314	0.0136	0.0684	0.0902
6.2	4445	3603	2652	1861	1270	0842	0531	0302	0131	0656	0859
6.4	4464	3594	2622	1825	1238	0816	0513	0291	0126	0629	0819
6.6	4482	3584	2590	1788	1205	0791	0495	0280	0121	0603	0781
6.8	4499	3572	2558	1751	1173	0767	0479	0270	0116	0579	0746
7.0	0.4515	0.3560	0.2526	0.1716	0.1142	0.0743	0.0462	0.0260	0.0112	0.0556	0.0712
7.2	4530	3546	2493	1680	1112	0720	0447	0251	0107	0535	0681
7.4	4544	3532	2460	1645	1082	0698	0432	0242	0103	0514	0651
7.6	4557	3516	2426	1610	1053	0677	0417	0233	0100	0494	0624
7.8	4570	3500	2393	1576	1026	0657	0404	0225	0096	0476	0597
8.0	0.4581	0.3483	0.2359	0.1542	0.0999	0.0637	0.0390	0.0218	0.0093	0.0458	0.0573
8 2	4592	·3466	2326	1510	0972	0618	0378	0210	0089	0441	0549
8.4	4602	3447	2293	1477	0946	0600	0366	0203	0086	0425	0527
8.6	4612	3429	2259	1446	0922	0582	0354	0196	0083	0410	0506
8.8	4622	3410	2227	1415	0898	0565	0343	0190	0080	0396	0487
9.0	0.4630	0.3390	0.2193	0.1384	0.0875	0.0549	0.0332	0.0184	0.0077	0.0382	0.0468
9.2	4639	3370	2161	1355	0852	0533	0322	0177	0075	0369	0450
9.4	4646	3350	2129	1326	0830	0518	0313	0172	0072	0356	0434
9.6	4653	3329	2096	1298	0809	0503	0303	0167	0070	0345	0418
9.8	4661	3308	2065	1270	0788	0489	0294	0161	0068	0333	0403
10.0	0.4667	0.3286	0.2033	0.1243	0.0768	0.0475	0.0285	0.0156	0.0066	0.0323	0.0388

Table 2.

x' positive $l' = 5$

x'	k'_t	k'_x	$C'_{\perp}\cos\varphi$								
			0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.1961	0.1868	0.1727	0.1547	0.1334	0.1096	0.0838	0.0566	0.0285
0.2	0.9608	0.1922	1885	1798	1667	1498	1298	1071	0823	0560	0284
0.4	8595	3437	1807	1728	1606	1449	1259	1044	0807	0550	0280
0.6	7303	4381	1732	1659	1545	1397	1220	1016	0789	0541	0277
0.8	6024	4819	1657	1589	1484	1347	1180	0986	0769	0531	0273
1.0	0.4906	0.4906	0.1584	0.1521	0.1424	0.1295	0.1138	0.0955	0.0748	0.0518	0.0268
1.2	3989	4787	1512	1454	1364	1244	1096	0924	0726	0505	0263
1.4	3257	4559	1442	1389	1305	1193	1055	0891	0704	0492	0257
1.6	2680	4288	1374	1324	1247	1143	1013	0859	0680	0478	0251
1.8	2224	4003	1308	1263	1191	1094	0973	0827	0657	0463	0244
2.0	0.1862	0.3724	0.1245	0.1203	0.1137	0.1045	0.0932	0.0795	0.0634	0.0449	0.0237
2.2	1572	3459	1184	1146	1083	0999	0892	0763	0611	0433	0230
2.4	1338	3212	1126	1089	1032	0954	0854	0732	0587	0418	0223
2.6	1148	2985	1070	1037	0984	0910	0817	0702	0565	0404	0216
2.8	0991	2776	1017	0987	0937	0869	0780	0672	0543	0388	0209
3.0	0.0862	0.2586	0.0967	0.0939	0.0892	0.0828	0.0746	0.0644	0.0521	0.0374	0.0202
3.2	0754	2411	0919	0893	0850	0789	0711	0616	0499	0360	0195
3.4	0662	2252	0873	0849	0809	0753	0680	0589	0478	0346	0187
3.6	0585	2107	0830	0808	0770	0717	0649	0564	0458	0332	0181
3.8	0520	1973	0789	0768	0734	0684	0620	0539	0439	0319	0174
4.0	0.0463	0.1851	0.0751	0.0732	0.0699	0.0652	0.0592	0.0515	0.0421	0.0306	0.0168
4.2	0414	1739	0715	0696	0666	0622	0565	0492	0403	0294	0161
4.4	0372	1636	0680	0663	0634	0594	0540	0471	0386	0282	0155
4.6	0335	1541	0647	0632	0605	0567	0515	0451	0370	0271	0149
4.8	0303	1454	0617	0602	0577	0541	0493	0431	0355	0260	0143
5.0	0.0275	0.1373	0.0588	0.0575	0.0551	0.0516	0.0471	0.0413	0.0340	0.0249	0.0138
5.2	0250	1298	0561	0548	0526	0494	0450	0395	0326	0239	0132
5.4	0228	1229	0536	0523	0503	0472	0431	0378	0312	0230	0127
5.6	0208	1165	0511	0500	0480	0451	0412	0362	0300	0221	0123
5.8	0191	1105	0488	0478	0459	0432	0395	0347	0287	0212	0118
6.0	0.0175	0.1050	0.0467	0.0457	0.0439	0.0413	0.0378	0.0333	0.0276	0.0204	0.0113
6.2	0161	0999	0447	0437	0420	0396	0363	0320	0265	0196	0109
6.4	0148	0950	0427	0419	0403	0379	0348	0307	0254	0188	0105
6.6	0137	0905	0409	0401	0386	0364	0334	0294	0245	0181	0101
6.8	0127	0863	0392	0385	0370	0349	0320	0283	0235	0175	0098
7.0	0.0118	0.0824	0.0376	0.0369	0.0355	0.0335	0.0307	0.0271	0.0226	0.0168	0.0094
7.2	0109	0788	0361	0354	0341	0322	0295	0261	0217	0162	0091
7.4	0102	0753	0346	0340	0327	0309	0284	0251	0209	0155	0087
7.6	0095	0720	0333	0326	0315	0297	0274	0242	0202	0150	0084
7.8	0089	0690	0320	0314	0303	0286	0263	0233	0194	0144	0082
8.0	0.0083	0.0662	0.0307	0.0302	0.0291	0.0275	0.0253	0.0225	0.0187	0.0140	0.0078
8.2	0078	0635	0296	0290	0280	0265	0244	0217	0181	0135	0076
8.4	0073	0609	0285	0280	0270	0255	0235	0209	0174	0130	0073
8.6	0068	0585	0274	0269	0260	0246	0226	0201	0168	0125	0071
8.8	0064	0563	0264	0260	0251	0237	0219	0194	0162	0121	0069
9.0	0.0060	0.0541	0.0255	0.0250	0.0242	0.0229	0.0211	0.0187	0.0157	0.0117	0.0067
9.2	0057	0521	0246	0242	0234	0221	0204	0181	0152	0113	0064
9.4	0053	0502	0237	0233	0226	0214	0197	0175	0147	0110	0062
9.6	0050	0483	0229	0225	0218	0206	0190	0169	0142	0106	0060
9.8	0048	0466	0222	0218	0210	0199	0184	0164	0137	0103	0058
10.0	0.0045	0.0450	0.0214	0.0211	0.0204	0.0193	0.0178	0.0159	0.0133	0.0099	0.0056

Table 2.

x′ negative $l' = 5$

$-x'$	$C'_{\perp}\cos\varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90°
0.0	0.1961	0.1868	0.1727	0.1547	0.1334	0.1096	0.0838	0.0566	0.0285	0.1640	0.2440
0.2	2038	1938	1787	1594	1369	1119	0851	0572	0287	1637	2435
0.4	2114	2006	1843	1639	1401	1139	0862	0576	0287	1630	2423
0.6	2190	2074	1900	1681	1431	1158	0871	0579	0287	1619	2402
0.8	2265	2140	1954	1722	1458	1173	0878	0580	0286	1603	2374
1.0	0.2338	0.2204	0.2005	0.1759	0.1482	0.1185	0.0882	0.0579	0.0284	0.1584	0.2339
1.2	2410	2266	2054	1794	1503	1195	0884	0577	0281	1560	2297
1.4	2480	2326	2100	1825	1521	1203	0884	0574	0278	1534	2249
1.6	2548	2383	2144	1853	1535	1207	0882	0569	0274	1504	2196
1.8	2614	2439	2184	1878	1546	1208	0878	0563	0270	1472	2140
2.0	0.2677	0.2490	0.2220	0.1899	0.1555	0.1207	0.0871	0.0556	0.0264	0.1437	0.2079
2.2	2738	2539	2253	1917	1559	1203	0863	0547	0259	1401	2016
2.4	2797	2586	2284	1932	1561	1197	0853	0537	0253	1364	1952
2.6	2852	2629	2311	1942	1560	1188	0842	0527	0248	1325	1886
2.8	2905	2670	2334	1949	1556	1178	0829	0517	0241	1286	1820
3.0	0.2955	0.2707	0.2355	0.1955	0.1550	0.1165	0.0816	0.0506	0.0234	0.1246	0.1753
3.2	3003	2742	2372	1957	1541	1151	0801	0494	0228	1207	1687
3.4	3049	2774	2387	1956	1529	1136	0785	0482	0221	1167	1622
3.6	3092	2803	2398	1952	1516	1118	0770	0470	0215	1128	1558
3.8	3133	2830	2407	1946	1502	1101	0753	0457	0208	1090	1513
4.0	0.3172	0.2854	0.2413	0.1938	0.1485	0.1082	0.0736	0.0445	0.0201	0.1052	0.1436
4.2	3208	2875	2417	1927	1467	1062	0719	0432	0195	1015	1377
4.4	3243	2895	2418	1915	1448	1042	0701	0419	0189	0979	1320
4.6	3275	2912	2417	1900	1427	1021	0683	0408	0183	0944	1266
4.8	3305	2927	2414	1884	1405	0999	0666	0395	0177	0910	1213
5.0	0.3334	0.2941	0.2409	0.1867	0.1383	0.0977	0.0648	0.0383	0.0171	0.0877	0.1163
5.2	3361	2952	2402	1849	1360	0956	0631	0372	0165	0845	1114
5.4	3387	2962	2394	1829	1337	0934	0614	0360	0159	0815	1068
5.6	3411	2970	2385	1808	1314	0913	0597	0349	0154	0785	1024
5.8	3434	2976	2373	1787	1289	0891	0580	0338	0149	0757	0982
6.0	0.3455	0.2981	0.2360	0.1765	0.1265	0.0870	0.0564	0.0328	0.0144	0.0730	0.0942
6.2	3476	2985	2347	1742	1241	0849	0548	0318	0139	0704	0904
6.4	3495	2988	2332	1719	1217	0828	0533	0307	0135	0679	0868
6.6	3513	2989	2316	1695	1193	0808	0518	0298	0130	0655	0833
6.8	3530	2989	2300	1670	1169	0788	0503	0289	0125	0632	0800
7.0	0.3546	0.2989	0.2282	0.1646	0.1145	0.0768	0.0488	0.0280	0.0121	0.0609	0.0769
7.2	3562	2987	2264	1621	1121	0749	0474	0271	0117	0588	0739
7.4	3576	2983	2245	1597	1097	0730	0461	0262	0113	0568	0710
7.6	3590	2980	2226	1572	1074	0711	0448	0254	0110	0549	0683
7.8	3603	2976	2206	1547	1052	0694	0436	0246	0106	0530	0658
8.0	0.3615	0.2970	0.2185	0.1522	0.1029	0.0676	0.0423	0.0239	0.0103	0.0513	0.0633
8.2	3627	2965	2165	1497	1007	0659	0411	0232	0100	0496	0610
8.4	3637	2958	2143	1473	0985	0642	0400	0225	0097	0479	0588
8.6	3648	2950	2122	1448	0964	0626	0389	0218	0093	0464	0567
8.8	3658	2942	2100	1424	0943	0610	0378	0212	0090	0449	0547
9.0	0.3667	0.2934	0.2078	0.1400	0.0923	0.0595	0.0367	0.0206	0.0088	0.0435	0.0528
9.2	3676	2925	2056	1376	0903	0580	0357	0199	0085	0421	0509
9.4	3685	2916	2034	1352	0883	0565	0348	0194	0082	0408	0492
9.6	3693	2906	2012	1329	0864	0552	0338	0188	0080	0395	0475
9.8	3701	2895	1989	1306	0845	0538	0329	0183	0078	0383	0459
10.0	0.3708	0.2885	0.1967	0.1284	0.0827	0.0525	0.0320	0.0178	0.0075	0.0372	0.0444

Table 2.

x' positive $l' = 6$

x'	k'_t	k'_x	$C'_\perp \cos\varphi$ 0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.1644	0.1574	0.1463	0.1316	0.1139	0.0939	0.0720	0.0487	0.0246
0.2	0.9610	0.1922	1590	1524	1419	1280	1112	0921	0709	0482	0245
0.4	8602	3441	1536	1475	1376	1244	1085	0901	0696	0476	0242
0.6	7317	4390	1483	1425	1333	1208	1057	0881	0684	0469	0240
0.8	6046	4836	1430	1376	1289	1172	1027	0860	0670	0461	0237
1.0	0.4934	0.4934	0.1378	0.1327	0.1245	0.1135	0.0999	0.0838	0.0655	0.0453	0.0234
1.2	4021	4825	1325	1278	1202	1098	0968	0815	0640	0443	0231
1.4	3292	4609	1276	1232	1160	1061	0938	0792	0624	0435	0227
1.6	2717	4346	1225	1185	1117	1024	0908	0769	0607	0425	0222
1.8	2261	4071	1178	1139	1076	0989	0878	0746	0591	0415	0218
2.0	0.1900	0.3800	0.1130	0.1094	0.1035	0.0953	0.0848	0.0722	0.0574	0.0405	0.0213
2.2	1610	3542	1085	1051	0995	0918	0819	0699	0558	0394	0209
2.4	1376	3302	1041	1009	0957	0884	0790	0675	0540	0383	0204
2.6	1185	3081	0998	0968	0919	0851	0761	0653	0523	0372	0199
2.8	1027	2877	0957	0929	0883	0818	0734	0630	0507	0362	0193
3.0	0.0897	0.2691	0.0917	0.0891	0.0848	0.0786	0.0707	0.0608	0.0491	0.0351	0.0188
3.2	0788	2520	0879	0855	0813	0756	0681	0587	0473	0340	0182
3.4	0695	2363	0841	0820	0780	0726	0655	0566	0457	0329	0177
3.6	0616	2219	0806	0786	0750	0697	0630	0545	0442	0318	0172
3.8	0550	2087	0773	0753	0719	0670	0606	0525	0427	0308	0167
4.0	0.0491	0.1966	0.0741	0.0722	0.0690	0.0643	0.0582	0.0505	0.0412	0.0298	0.0162
4.2	0442	1854	0710	0692	0662	0618	0560	0487	0397	0288	0157
4.4	0398	1751	0681	0664	0635	0594	0539	0469	0383	0279	0152
4.6	0360	1656	0652	0637	0609	0570	0518	0451	0369	0269	0148
4.8	0327	1568	0626	0611	0586	0548	0498	0435	0356	0260	0143
5.0	0.0297	0.1486	0.0600	0.0586	0.0562	0.0527	0.0479	0.0419	0.0344	0.0251	0.0138
5.2	0271	1409	0575	0563	0540	0506	0461	0404	0331	0242	0134
5.4	0248	1339	0553	0541	0518	0487	0443	0389	0319	0234	0129
5.6	0227	1273	0530	0519	0498	0468	0427	0374	0308	0226	0125
5.8	0209	1212	0510	0499	0479	0450	0411	0360	0297	0219	0121
6.0	0.0192	0.1154	0.0490	0.0479	0.0461	0.0433	0.0396	0.0347	0.0287	0.0211	0.0117
6.2	0178	1101	0471	0461	0443	0417	0381	0335	0277	0204	0113
6.4	0164	1051	0453	0443	0426	0401	0367	0323	0267	0197	0110
6.6	0152	1004	0435	0426	0410	0386	0354	0312	0258	0190	0106
6.8	0141	0959	0419	0411	0395	0372	0341	0301	0249	0184	0103
7.0	0.0131	0.0918	0.0403	0.0396	0.0380	0.0359	0.0329	0.0290	0.0240	0.0178	0.0099
7.2	0122	0880	0388	0381	0367	0346	0317	0280	0232	0172	0096
7.4	0114	0843	0374	0367	0354	0333	0306	0271	0225	0166	0091
7.6	0106	0808	0361	0354	0341	0322	0296	0261	0217	0161	0089
7.8	0100	0795	0347	0341	0329	0311	0285	0252	0210	0156	0087
8.0	0.0093	0.0745	0.0335	0.0329	0.0318	0.0300	0.0275	0.0244	0.0203	0.0151	0.0085
8.2	0088	0716	0324	0318	0307	0290	0266	0236	0197	0146	0082
8.4	0082	0689	0313	0307	0296	0280	0257	0228	0190	0141	0080
8.6	0077	0662	0302	0296	0286	0270	0249	0221	0184	0137	0077
8.8	0073	0638	0292	0287	0276	0262	0241	0213	0178	0132	0074
9.0	0.0068	0.0615	0.0282	0.0277	0.0268	0.0253	0.0233	0.0207	0.0172	0.0129	0.0072
9.2	0065	0592	0272	0268	0259	0245	0226	0200	0167	0125	0070
9.4	0061	0572	0264	0259	0250	0237	0218	0194	0162	0121	0068
9.6	0057	0551	0255	0251	0242	0230	0212	0188	0157	0117	0066
9.8	0054	0533	0247	0243	0235	0223	0205	0182	0152	0114	0064
10.0	0.0051	0.0514	0.0240	0.0235	0.0228	0.0216	0.0199	0.0177	0.0148	0.0111	0.0062

Table 2.

x' negative $l' = 6$

$-x'$	$C'_{\perp} \cos \varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90°
0.0	0.1644	0.1574	0.1463	0.1316	0.1139	0.0939	0.0720	0.0487	0.0246	0.1412	0.1967
0.2	1698	1623	1505	1349	1164	0955	0730	0491	0247	1410	1965
0.4	1752	1672	1547	1383	1188	0971	0737	0495	0247	1406	1958
0.6	1805	1721	1587	1414	1211	0985	0745	0498	0247	1399	1947
0.8	1858	1769	1627	1445	1232	0997	0751	0499	0247	1390	1931
1.0	0.1910	0.1816	0.1666	0.1474	0.1251	0.1008	0.0756	0.0499	0.0246	0.1377	0.1911
1.2	1962	1861	1703	1501	1269	1018	0758	0498	0244	1363	1887
1.4	2012	1905	1739	1526	1284	1025	0760	0497	0243	1346	1859
1.6	2062	1949	1772	1550	1298	1031	0761	0495	0240	1327	1828
1.8	2111	1991	1805	1572	1310	1036	0759	0492	0238	1306	1795
2.0	0.2157	0.2031	0.1834	0.1591	0.1320	0.1038	0.0757	0.0488	0.0235	0.1283	0.1758
2.2	2203	2069	1863	1608	1328	1038	0754	0483	0231	1259	1720
2.4	2247	2106	1889	1624	1334	1037	0750	0478	0227	1234	1679
2.6	2291	2142	1914	1638	1338	1035	0744	0472	0223	1208	1638
2.8	2332	2176	1937	1649	1340	1031	0737	0465	0220	1180	1595
3.0	0.2371	0.2207	0.1958	0.1658	0.1340	0.1026	0.0729	0.0458	0.0215	0.1152	0.1551
3.2	2409	2237	1976	1666	1339	1019	0720	0450	0211	1124	1507
3.4	2446	2266	1993	1671	1336	1011	0711	0443	0206	1095	1463
3.6	2481	2293	2008	1675	1331	1002	0701	0435	0201	1067	1418
3.8	2515	2318	2021	1677	1325	0992	0691	0426	0197	1038	1374
4.0	0.2548	0.2341	0.2032	0.1677	0.1318	0.0981	0.0680	0.0417	0.0191	0.1009	0.1330
4.2	2578	2363	2042	1675	1308	0969	0668	0408	0187	0981	1287
4.4	2607	2384	2050	1672	1299	0956	0656	0399	0182	0952	1245
4.6	2635	2402	2056	1668	1287	0944	0644	0390	0177	0925	1204
4.8	2663	2419	2061	1662	1276	0929	0632	0381	0173	0897	1163
5.0	0.2688	0.2436	0.2063	0.1655	0.1263	0.0915	0.0619	0.0372	0.0168	0.0871	0.1124
5.2	2712	2450	2065	1646	1249	0901	0607	0363	0163	0844	1085
5.4	2736	2463	2066	1636	1235	0885	0594	0354	0159	0819	1048
5.6	2757	2476	2065	1626	1220	0870	0580	0345	0154	0794	1012
5.8	2778	2487	2063	1615	1204	0855	0568	0337	0150	0770	0977
6.0	0.2799	0.2496	0.2060	0.1603	0.1188	0.0839	0.0555	0.0327	0.0146	0.0746	0.0944
6.2	2817	2505	2055	1589	1172	0824	0543	0319	0141	0723	0911
6.4	2836	2512	2050	1576	1155	0808	0530	0311	0138	0701	0880
6.6	2853	2519	2044	1561	1138	0792	0518	0303	0134	0679	0850
6.8	2870	2525	2037	1546	1121	0777	0506	0295	0130	0659	0821
7.0	0.2885	0.2530	0.2029	0.1530	0.1103	0.0761	0.0494	0.0287	0.0126	0.0638	0.0793
7.2	2900	2534	2021	1515	1086	0745	0482	0279	0122	0619	0766
7.4	2914	2537	2011	1498	1068	0730	0471	0272	0119	0600	0740
7.6	2928	2539	2001	1482	1051	0715	0459	0265	0115	0582	0715
7.8	2940	2541	1990	1465	1033	0700	0448	0258	0112	0565	0692
8.0	0.2952	0.2541	0.1979	0.1447	0.1016	0.0686	0.0438	0.0251	0.0109	0.0548	0.0669
8.2	2965	2542	1967	1430	0999	0671	0427	0244	0106	0531	0647
8.4	2975	2542	1955	1412	0981	0657	0417	0238	0103	0516	0626
8.6	2986	2541	1942	1395	0964	0643	0407	0232	0100	0501	0606
8.8	2997	2539	1929	1377	0947	0629	0397	0225	0097	0486	0586
9.0	0.3006	0.2538	0.1915	0.1359	0.0930	0.0616	0.0387	0.0220	0.0095	0.0472	0.0568
9.2	3015	2535	1901	1341	0914	0603	0378	0214	0092	0458	0550
9.4	3026	2532	1888	1325	0898	0590	0369	0208	0090	0446	0533
9.6	3033	2529	1873	1305	0881	0577	0360	0203	0087	0433	0516
9.8	3041	2525	1858	1287	0865	0565	0352	0198	0085	0421	0501
10.0	0.3049	0.2521	0.1843	0.1270	0.0849	0.0553	0.0343	0.0193	0.0083	0.0409	0.0485

Table 2.

x′ positive $l' = 8$

x'	k'_t	k'_x	$C_{,\perp} \cos \varphi$								
			0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.1240	0.1196	0.1118	0.1011	0.0880	0.0728	0.0560	0.0380	0.0192
0.2	0.9612	0.1922	1210	1167	1093	0991	0865	0717	0554	0377	0191
0.4	8610	3444	1179	1139	1068	0970	0848	0706	0546	0374	0190
0.6	7333	4399	1148	1110	1042	0948	0832	0694	0540	0370	0188
0.8	6068	4854	1118	1082	1017	0927	0815	0682	0531	0365	0187
1.0	0.4962	0.4962	0.1088	0.1053	0.0991	0.0906	0.0798	0.0669	0.0523	0.0360	0.0186
1.2	4053	4864	1057	1025	0966	0884	0780	0656	0514	0356	0184
1.4	3328	4659	1029	0997	0941	0862	0763	0643	0505	0351	0182
1.6	2755	4408	0999	0969	0916	0841	0744	0630	0496	0345	0179
1.8	2301	4143	0970	0942	0891	0819	0727	0616	0487	0340	0178
2.0	0.1941	0.3882	0.0941	0.0914	0.0867	0.0798	0.0709	0.0603	0.0478	0.0334	0.0175
2.2	1652	3634	0914	0889	0842	0777	0692	0588	0468	0328	0173
2.4	1418	3402	0887	0862	0818	0755	0674	0575	0458	0322	0170
2.6	1227	3189	0859	0837	0795	0735	0657	0561	0447	0316	0168
2.8	1068	2992	0833	0811	0771	0714	0640	0547	0437	0310	0164
3.0	0.0937	0.2812	0.0808	0.0787	0.0749	0.0694	0.0622	0.0533	0.0427	0.0304	0.0161
3.2	0827	2646	0783	0763	0727	0674	0605	0520	0417	0296	0158
3.4	0733	2494	0758	0739	0605	0654	0589	0506	0407	0290	0155
3.6	0654	2355	0735	0717	0683	0636	0572	0493	0397	0284	0152
3.8	0586	2226	0712	0695	0663	0617	0555	0479	0387	0277	0149
4.0	0.0527	0.2108	0.0689	0.0673	0.0642	0.0598	0.0540	0.0467	0.0377	0.0271	0.0146
4.2	0476	1998	0667	0652	0623	0581	0524	0454	0367	0264	0143
4.4	0431	1897	0646	0631	0604	0564	0509	0442	0357	0258	0140
4.6	0392	1803	0625	0611	0585	0546	0495	0429	0348	0252	0136
4.8	0357	1716	0606	0592	0567	0530	0480	0418	0339	0246	0133
5.0	0.0327	0.1634	0.0586	0.0574	0.0550	0.0514	0.0467	0.0405	0.0331	0.0240	0.0130
5.2	0300	1558	0568	0555	0533	0498	0453	0394	0322	0234	0127
5.4	0276	1487	0550	0538	0516	0484	0440	0383	0313	0228	0124
5.6	0254	1421	0533	0522	0500	0469	0426	0372	0305	0222	0121
5.8	0235	1359	0516	0506	0485	0455	0414	0361	0296	0216	0119
6.0	0.0217	0.1300	0.0500	0.0489	0.0470	0.0441	0.0402	0.0351	0.0288	0.0210	0.0116
6.2	0201	1246	0484	0475	0456	0428	0390	0342	0280	0205	0113
6.4	0186	1194	0469	0460	0442	0415	0378	0332	0272	0200	0110
6.6	0173	1146	0455	0446	0429	0402	0368	0323	0265	0194	0108
6.8	0162	1100	0440	0432	0416	0391	0357	0313	0258	0190	0105
7.0	0.0151	0.1057	0.0427	0.0419	0.0403	0.0379	0.0347	0.0304	0.0251	0.0184	0.0103
7.2	1041	1017	0414	0407	0391	0368	0336	0296	0244	0180	0099
7.4	0132	0978	0402	0394	0379	0357	0327	0287	0237	0175	0097
7.6	0124	0941	0389	0382	0368	0347	0318	0279	0231	0170	0095
7.8	0116	0906	0378	0371	0357	0337	0309	0272	0225	0166	0092
8.0	0.0109	0.0874	0.0367	0.0360	0.0347	0.0327	0.0300	0.0264	0.0219	0.0162	0.0089
8.2	0103	0843	0356	0349	0337	0318	0292	0257	0213	0157	0087
8.4	0097	0813	0346	0339	0328	0309	0283	0250	0207	0153	0085
8.6	0091	0785	0335	0330	0318	0300	0276	0243	0202	0149	0083
8.8	0086	0758	0326	0320	0309	0292	0268	0236	0197	0145	0081
9.0	0.0081	0.0733	0.0316	0.0311	0.0300	0.0284	0.0260	0.0230	0.0192	0.0141	0.0079
9.2	0077	0708	0308	0303	0292	0276	0253	0224	0187	0138	0077
9.4	0073	0685	0299	0294	0284	0268	0247	0218	0182	0134	0076
9.6	0069	0662	0291	0286	0276	0261	0240	0212	0177	0131	0073
9.8	0065	0642	0283	0278	0268	0254	0234	0206	0172	0128	0072
10.0	0.0062	0.0621	0.0275	0.0271	0.0261	0.0247	0.0227	0.0201	0.0168	0.0125	0.0070

Table 2.

x' negative $l' = 8$

$-x'$	$C'_{\perp} \cos \varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	− 90°
0.0	0.1240	0.1196	0.1118	0.1011	0.0880	0.0728	0.0560	0.0380	0.0192	0.1103	0.1416
0.2	1272	1225	1143	1032	0896	0739	0566	0383	0193	1103	1415
0.4	1301	1252	1168	1052	0910	0748	0571	0386	0193	1101	1412
0.6	1332	1280	1192	1071	0924	0757	0576	0387	0193	1098	1408
0.8	1363	1309	1215	1090	0937	0765	0581	0388	0193	1093	1401
1.0	0.1393	0.1337	0.1240	0.1108	0.0950	0.0773	0.0585	0.0389	0.0193	0.1088	0.1393
1.2	1423	1364	1262	1126	0962	0780	0588	0390	0193	1081	1383
1.4	1452	1390	1284	1142	0973	0787	0590	0390	0192	1073	1372
1.6	1482	1417	1306	1158	0985	0793	0591	0389	0191	1064	1359
1.8	1511	1442	1327	1174	0994	0797	0593	0388	0189	1054	1344
2.0	0.1539	0.1468	0.1347	0.1188	0.1003	0.0801	0.0594	0.0387	0.0188	0.1043	0.1328
2.2	1568	1492	1367	1202	1010	0805	0594	0386	0187	1031	1311
2.4	1594	1517	1386	1214	1018	0807	0593	0384	0185	1018	1293
2.6	1621	1541	1404	1227	1024	0809	0592	0382	0184	1005	1274
2.8	1647	1563	1422	1238	1027	0810	0590	0379	0181	0991	1254
3.0	0.1673	0.1586	0.1438	0.1247	0.1034	0.0810	0.0588	0.0376	0.0179	0.0976	0.1233
3.2	1698	1607	1454	1258	1037	0810	0586	0373	0178	0961	1211
3.4	1722	1627	1469	1265	1039	0808	0582	0369	0175	0945	1189
3.6	1746	1647	1482	1273	1042	0806	0578	0366	0173	0929	1166
3.8	1769	1667	1495	1280	1043	0804	0574	0362	0170	0913	1143
4.0	0.1792	0.1685	0.1508	0.1285	0.1043	0.0800	0.0570	0.0357	0.0168	0.0896	0.1120
4.2	1814	1703	1519	1290	1042	0796	0564	0353	0165	0879	1096
4.4	1835	1720	1530	1294	1041	0792	0559	0348	0162	0862	1072
4.6	1855	1736	1539	1298	1040	0787	0553	0343	0159	0845	1049
4.8	1875	1752	1548	1299	1037	0782	0547	0339	0156	0828	1025
5.0	0.1894	0.1767	0.1557	0.1301	0.1033	0.0776	0.0541	0.0333	0.0154	0.0810	0.1001
5.2	1913	1782	1504	1302	1029	0770	0534	0329	0151	0793	0978
5.4	1931	1795	1570	1302	1025	0763	0528	0323	0148	0776	0955
5.6	1948	1808	1576	1301	1020	0757	0521	0318	0145	0759	0932
5.8	1965	1819	1581	1300	1014	0749	0514	0313	0142	0743	0909
6.0	0.1981	0.1831	0.1586	0.1298	0.1008	0.0741	0.0507	0.0307	0.0140	0.0726	0.0886
6.2	1996	1842	1589	1295	1001	0733	0499	0301	0136	0710	0864
6.4	2011	1852	1592	1292	0995	0725	0492	0296	0134	0694	0843
6.6	2026	1862	1595	1288	0987	0717	0484	0291	0131	0678	0822
6.8	2040	1871	1597	1284	0979	0708	0477	0285	0128	0662	0801
7.0	0.2053	0.1880	0.1598	0.1278	0.0971	0.0699	0.0470	0.0280	0.0125	0.0647	0.0780
7.2	2067	1887	1598	1273	0962	0690	0462	0275	0123	0632	0760
7.4	2079	1895	1598	1267	0954	0681	0455	0270	0120	0617	0741
7.6	2091	1901	1597	1261	0945	0672	0447	0264	0118	0602	0722
7.8	2103	1908	1596	1254	0935	0663	0439	0259	0115	0588	0703
8.0	0.2114	0.1914	0.1595	0.1247	0.0926	0.0654	0.0432	0.0254	0.0113	0.0574	0.0685
8.2	2125	1919	1593	1240	0916	0644	0424	0249	0111	0561	0668
8.4	2135	1924	1591	1232	0906	0635	0417	0244	0107	0548	0651
8.6	2146	1929	1587	1224	0897	0626	0410	0239	0105	0435	0634
8.8	2155	1933	1584	1215	0887	0616	0402	0234	0103	0522	0618
9.0	0.2165	0.1936	0.1580	0.1206	0.0877	0.0607	0.0395	0.0228	0.0101	0.0510	0.0602
9.2	2173	1940	1576	1198	0866	0598	0388	0225	0098	0498	0587
9.4	2182	1943	1572	1189	0856	0589	0381	0220	0096	0486	0572
9.6	2191	1946	1567	1180	0846	0579	0374	0216	0094	0475	0558
9.8	2198	1947	1562	1170	0835	0571	0367	0212	0092	0464	0543
10.0	0.2206	0.1950	0.1556	0.1161	0.0825	0.0562	0.0360	0.0207	0.0090	0.0453	0.0530

Table 2.

x' positive $l' = 10$

x'	k'_t	k'_x	$C'_{\perp} \cos\varphi$ 0°	10°	20°	30°	40°	50°	60°	70°	80°
0.0	1.0000	0	0.0995	0.0963	0.0904	0.0821	0.0716	0.0594	0.0458	0.0311	0.0157
0.2	0.9613	0.1923	0975	0945	0888	0807	0706	0587	0454	0309	0157
0.4	8614	3445	0955	0926	0871	0793	0695	0580	0445	0307	0156
0.6	7340	4403	0935	0907	0854	0780	0685	0572	0444	0304	0156
0.8	6079	4863	0916	0889	0838	0766	0674	0564	0440	0302	0155
1.0	0.4975	0.4975	0.0897	0.0871	0.0822	0.0751	0.0663	0.0556	0.0435	0.0300	0.0153
1.2	4069	4883	0877	0852	0805	0738	0651	0548	0429	0296	0152
1.4	3346	4684	0858	0834	0788	0723	0640	0540	0423	0293	0151
1.6	2774	4438	0838	0816	0772	0709	0628	0530	0417	0289	0150
1.8	2321	4179	0820	0791	0755	0695	0616	0522	0410	0285	0149
2.0	0.1962	0.3923	0.0801	0.0780	0.0740	0.0681	0.0604	0.0513	0.0405	0.0282	0.0147
2.2	1672	3680	0782	0762	0723	0666	0593	0504	0399	0278	0146
2.4	1439	3453	0764	0744	0707	0653	0582	0494	0392	0275	0144
2.6	1248	3244	0745	0727	0691	0638	0570	0485	0385	0271	0142
2.8	1090	3051	0728	0710	0675	0624	0558	0477	0380	0267	0141
3.0	0.0958	0.2875	0.0710	0.0693	0.0659	0.0610	0.0546	0.0468	0.0373	0.0263	0.0139
3.2	0849	2716	0694	0677	0645	0598	0536	0459	0367	0259	0137
3.4	0754	2564	0676	0660	0629	0584	0524	0449	0359	0255	0135
3.6	0674	2428	0660	0644	0615	0571	0512	0440	0353	0250	0133
3.8	0606	2302	0644	0629	0600	0557	0501	0431	0346	0246	0131
4.0	0.0546	0.2186	0.0627	0.0613	0.0585	0.0544	0.0490	0.0422	0.0339	0.0242	0.0129
4.2	0495	2079	0612	0598	0571	0531	0479	0413	0333	0238	0127
4.4	0450	1980	0596	0583	0557	0518	0468	0404	0326	0234	0125
4.6	0410	1888	0581	0568	0544	0506	0457	0396	0320	0229	0124
4.8	0375	1802	0566	0554	0531	0494	0447	0387	0314	0226	0121
5.0	0.0345	0.1722	0.0552	0.0540	0.0517	0.0483	0.0436	0.0379	0.0307	0.0221	0.0119
5.2	0317	1647	0538	0527	0505	0471	0426	0370	0301	0217	0116
5.4	0292	1577	0524	0514	0492	0460	0417	0362	0294	0212	0115
5.6	0270	1511	0511	0501	0480	0449	0407	0354	0288	0208	0113
5.8	0250	1450	0498	0488	0468	0438	0398	0346	0281	0204	0111
6.0	0.0232	0.1392	0.0485	0.0476	0.0456	0.0428	0.0388	0.0338	0.0276	0.0200	0.0108
6.2	0216	1338	0472	0463	0445	0417	0379	0330	0270	0196	0106
6.4	0201	1286	0461	0451	0434	0407	0369	0323	0264	0192	0105
6.6	0187	1238	0449	0441	0423	0397	0361	0315	0258	0188	0103
6.8	0176	1192	0437	0429	0412	0388	0353	0308	0253	0184	0100
7.0	0.0164	0.1149	0.0427	0.0418	0.0402	0.0377	0.0345	0.0301	0.0247	0.0181	0.0098
7.2	0154	1108	0415	0408	0392	0368	0337	0294	0241	0176	0097
7.4	0144	1069	0405	0398	0382	0360	0333	0287	0236	0172	0095
7.6	0136	1031	0395	0388	0372	0351	0321	0281	0231	0169	0093
7.8	0128	0996	0385	0378	0364	0342	0313	0274	0225	0165	0092
8.0	0.0120	0.0963	0.0376	0.0369	0.0355	0.0334	0.0305	0.0268	0.0221	0.0162	0.0089
8.2	0114	0931	0366	0360	0346	0326	0298	0262	0215	0158	0087
8.4	0107	0901	0357	0351	0338	0318	0291	0256	0211	0155	0085
8.6	0101	0871	0348	0342	0330	0311	0284	0250	0206	0152	0084
8.8	0096	0844	0339	0334	0322	0303	0278	0245	0202	0149	0082
9.0	0.0091	0.0818	0.0332	0.0326	0.0314	0.0296	0.0271	0.0239	0.0197	0.0145	0.0080
9.2	0086	0792	0323	0318	0307	0289	0265	0234	0193	0142	0079
9.4	0082	0768	0316	0310	0299	0283	0259	0229	0189	0139	0077
9.6	0077	0745	0308	0303	0292	0276	0253	0223	0185	0137	0076
9.8	0074	0723	0301	0296	0285	0269	0247	0218	0181	0133	0074
10.0	0.0070	0.0702	0.0293	0.0289	0.0279	0.0263	0.0242	0.0213	0.0177	0.0130	0.0073

Table 2.

x' negative $l' = 10$

$-x'$	$C'_{\perp} \cos \varphi$									$C'_{\perp}$	
	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	−90°
0.0	0.0995	0.0963	0.0904	0.0821	0.0716	0.0594	0.0458	0.0311	0.0157	0.0905	0.1105
0.2	1015	0982	0921	0834	0727	0601	0462	0313	0158	0905	1105
0.4	1034	1000	0936	0847	0736	0608	0466	0315	0158	0904	1103
0.6	1054	1018	0952	0860	0746	0614	0469	0316	0159	0902	1101
0.8	1074	1037	0969	0873	0755	0620	0473	0317	0158	0900	1098
1.0	0.1094	0.1055	0.0984	0.0886	0.0764	0.0626	0.0476	0.0318	0.0158	0.0897	0.1094
1.2	1113	1073	1000	0898	0773	0632	0479	0319	0158	0893	1089
1.4	1132	1090	1014	0909	0780	0636	0480	0320	0158	0889	1083
1.6	1152	1108	1029	0920	0788	0640	0482	0320	0158	0884	1076
1.8	1170	1126	1044	0931	0795	0645	0483	0320	0157	0878	1069
2.0	0.1190	0.1143	0.1058	0.0943	0.0803	0.0648	0.0485	0.0320	0.0157	0.0872	0.1061
2.2	1207	1160	1072	0952	0810	0652	0486	0319	0155	0865	1052
2.4	1227	1176	1085	0962	0815	0655	0486	0318	0155	0858	1042
2.6	1244	1193	1099	0972	0821	0657	0487	0318	0154	0851	1032
2.8	1262	1209	1112	0981	0827	0659	0486	0316	0152	0843	1021
3.0	0.1279	0.1225	0.1124	0.0989	0.0831	0.0660	0.0486	0.0315	0.0151	0.0834	0.1010
3.2	1300	1243	1139	0998	0837	0663	0486	0314	0151	0827	1000
3.4	1314	1255	1147	1005	0839	0662	0484	0312	0150	0816	0986
3.6	1330	1269	1159	1012	0843	0662	0483	0310	0148	0806	0973
3.8	1347	1283	1169	1019	0846	0663	0482	0308	0147	0796	0960
4.0	0.1363	0.1298	0.1180	0.1025	0.0847	0.0662	0.0480	0.0306	0.0145	0.0786	0.0946
4.2	1378	1311	1190	1031	0850	0662	0478	0304	0144	0776	0933
4.4	1394	1325	1200	1036	0851	0660	0475	0301	0142	0765	0919
4.6	1409	1338	1208	1040	0852	0658	0473	0298	0141	0754	0905
4.8	1424	1350	1217	1045	0853	0658	0470	0296	0139	0743	0890
5.0	0.1438	0.1362	0.1225	0.1048	0.0853	0.0655	0.0466	0.0293	0.0137	0.0732	0.0876
5.2	1452	1374	1233	1053	0854	0654	0464	0290	0135	0721	0861
5.4	1466	1385	1239	1054	0853	0651	0460	0287	0134	0710	0846
5.6	1480	1396	1247	1057	0852	0647	0456	0284	0132	0699	0832
5.8	1492	1407	1253	1059	0851	0644	0452	0281	0130	0687	0817
6.0	0.1505	0.1417	0.1259	0.1061	0.0848	0.0641	0.0448	0.0277	0.0128	0.0676	0.0802
6.2	1517	1426	1265	1062	0847	0637	0444	0274	0126	0664	0787
6.4	1529	1436	1269	1063	0844	0633	0440	0271	0124	0653	0773
6.6	1541	1445	1275	1063	0842	0629	0435	0267	0122	0642	0758
6.8	1553	1454	1279	1063	0838	0624	0432	0263	0120	0631	0744
7.0	0.1564	0.1462	0.1283	0.1063	0.0835	0.0620	0.0427	0.0260	0.0119	0.0619	0.0730
7.2	1575	1470	1286	1062	0832	0615	0422	0257	0117	0608	0715
7.4	1585	1478	1289	1061	0828	0610	0417	0254	0115	0597	0701
7.6	1595	1485	1292	1060	0823	0605	0412	0250	0113	0586	0688
7.8	1605	1492	1295	1058	0819	0600	0408	0246	0111	0576	0674
8.0	0.1615	0.1499	0.1296	0.1056	0.0814	0.0594	0.0403	0.0243	0.0110	0.0565	0.0661
8.2	1624	1505	1299	1054	0810	0588	0398	0239	0108	0555	0647
8.4	1634	1512	1300	1051	0805	0582	0393	0235	0106	0544	0634
8.6	1642	1517	1301	1048	0799	0577	0388	0232	0104	0534	0621
8.8	1651	1523	1302	1045	0794	0571	0382	0228	0102	0524	0609
9.0	0.1659	0.1528	0.1302	0.1041	0.0788	0.0565	0.0378	0.0224	0.0100	0.0514	0.0596
9.2	1667	1523	1303	1037	0783	0559	0373	0221	0098	0504	0584
9.4	1675	1537	1302	1033	0777	0554	0368	0218	0096	0495	0572
9.6	1683	1542	1302	1029	0771	0547	0363	0214	0095	0485	0561
9.8	1690	1546	1301	1025	0765	0541	0358	0210	0093	0476	0549
10.0	0.1697	0.1550	0.1300	0.1020	0.0758	0.0535	0.0353	0.0207	0.0091	0.0467	0.0538

Table 3.

x'' positive $t = 0$

x''	k''_x	$C''_{\perp} \cdot \cos\varphi$								
		0°	± 10°	± 20°	± 30°	± 40°	± 50°	± 60°	± 70°	± 80°
0.0	∞	1.0000	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1736
0.2	4.9029	0.8039	7936	7629	7118	6404	5489	4378	3082	1615
0.4	2.3212	6286	6216	6005	5649	5141	4473	3635	2616	1406
0.6	1.4292	4855	4806	4658	4405	4040	3552	2924	2139	1172
0.8	0.9761	3753	3718	3610	3427	3159	2797	2323	1718	0955
1.0	0.7071	0.2929	0.2903	0.2823	0.2686	0.2484	0.2210	0.1847	0.1377	0.0773
1.2	5335	2318	2298	2237	2131	1976	1764	1481	1110	0628
1.4	4152	1863	1847	1799	1716	1594	1426	1201	0904	0514
1.6	3312	1520	1507	1469	1403	1304	1169	0986	0745	0425
1.8	2698	1258	1248	1217	1162	1082	0971	0821	0621	0356
2.0	0.2236	0.1056	0.1047	0.1021	0.0976	0.0909	0.0816	0.0691	0.0524	0.0301
2.2	1881	0896	0889	0867	0829	0773	0694	0588	0447	0257
2.4	1603	0769	0763	0744	0712	0664	0597	0506	0385	0221
2.6	1381	0666	0661	0645	0617	0576	0518	0439	0334	0193
2.8	1201	0583	0578	0564	0540	0503	0453	0385	0293	0169
3.0	0.1054	0.0513	0.0509	0.0497	0.0475	0.0444	0.0399	0.0339	0.0258	0.0149
3.2	0932	0455	0452	0441	0422	0394	0354	0301	0229	0133
3.4	0830	0406	0403	0393	0377	0352	0317	0269	0205	0119
3.6	0743	0365	0362	0353	0338	0316	0284	0242	0184	0107
3.8	0670	0329	0327	0319	0305	0285	0257	0218	0166	0096
4.0	0.0606	0.0297	0.0296	0.0289	0.0277	0.0259	0.0233	0.0198	0.0151	0.0087
4.2	0551	0272	0270	0263	0252	0235	0212	0180	0138	0080
4.4	0504	0249	0247	0241	0231	0215	0194	0165	0126	0073
4.6	0462	0228	0226	0221	0212	0198	0178	0152	0116	0067
4.8	0425	0210	0209	0204	0195	0182	0164	0140	0107	0062
5.0	0.0392	0.0194	0.0193	0.0188	0.0180	0.0168	0.0152	0.0129	0.0099	0.0057
5.2	0363	0180	0179	0174	0167	0156	0141	0120	0091	0053
5.4	0337	0167	0166	0162	0155	0145	0131	0111	0085	0049
5.6	0314	0156	0155	0151	0144	0135	0122	0104	0079	0046
5.8	0293	0145	0144	0141	0135	0126	0114	0097	0074	0043
6.0	0.0274	0.0136	0.0135	0.0132	0.0126	0.0118	0.0106	0.0091	0.0069	0.0040
6.2	0257	0128	0127	0124	0118	0111	0100	0085	0065	0037
6.4	0241	0120	0119	0116	0111	0104	0094	0080	0061	0035
6.6	0227	0113	0112	0109	0105	0098	0088	0075	0057	0033
6.8	0214	0106	0106	0103	0099	0092	0083	0071	0054	0031
7.0	0.0202	0.0100	0.0100	0.0097	0.0093	0.0087	0.0078	0.0067	0.0051	0.0029
7.2	0191	0095	0095	0092	0088	0082	0074	0063	0048	0027
7.4	0181	0090	0090	0087	0083	0078	0070	0060	0045	0026
7.6	0172	0085	0085	0083	0079	0074	0066	0057	0043	0025
7.8	0163	0081	0081	0079	0075	0070	0063	0054	0041	0024
8.0	0.0155	0.0077	0.0077	0.0075	0.0071	0.0067	0.0060	0.0051	0.0039	0.0023
8.2	0148	0073	0073	0071	0068	0064	0057	0049	0037	0022
8.4	0141	0070	0070	0068	0065	0061	0054	0047	0035	0021
8.6	0134	0067	0066	0065	0062	0058	0052	0045	0034	0020
8.8	0128	0064	0063	0062	0059	0055	0050	0043	0033	0019
9.0	0.0123	0.0061	0.0060	0.0059	0.0056	0.0053	0.0048	0.0041	0.0031	0.0018
9.2	0117	0058	0057	0056	0054	0051	0046	0039	0030	0017
9.4	0112	0056	0055	0054	0052	0049	0044	0037	0029	0016
9.6	0108	0054	0053	0052	0050	0047	0042	0035	0027	0016
9.8	0104	0052	0051	0050	0048	0045	0040	0034	0026	0015
10.0	0.0100	0.0050	0.0049	0.0048	0.0046	0.0043	0.0039	0.0033	0.0025	0.0015

Table 3.

x'' negative $t = 0$

$-x''$	$C''_{\perp} \cdot \cos\varphi$									$C''_{\perp}$
	0°	± 10°	± 20°	± 30°	± 40°	± 50°	± 60°	± 70°	± 80°	± 90°
0.0	1.0000	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1736	1.0000
0.2	1961	1.1736	1.1077	1.0031	8668	7072	5330	3525	1729	0.9615
0.4	3714	3385	2443	1006	9230	7279	5293	3377	1600	8621
0.6	5145	4679	3377	1485	9297	7062	4950	3052	1402	7353
0.8	6247	5605	3874	1505	8957	6549	4434	2652	1188	6098
1.0	1.7071	1.6217	1.4003	1.1171	0.8357	0.5892	0.3867	0.2256	0.0990	0.5000
1.2	7682	6579	3848	0605	7629	5204	3327	1901	0821	4098
1.4	8138	6750	3489	0.9908	6871	4553	2848	1601	0683	3378
1.6	8480	6778	2994	9159	6141	3969	2438	1353	0572	2809
1.8	8742	6695	2412	8407	5469	3460	2095	1151	0483	2358
2.0	1.8944	1.6529	1.1782	0.7684	0.4950	0.3024	0.1809	0.0986	0.0411	0.2000
2.2	9104	6299	1132	7008	4334	2653	1571	0850	0353	1712
2.4	9231	6019	0484	6387	3869	2338	1374	0739	0306	1479
2.6	9334	5689	0.9850	5822	3464	2071	1208	0647	0267	1289
2.8	9417	5352	9239	5312	3111	1843	1069	0571	0235	1131
3.0	1.9487	1.4983	0.8658	0.4854	0.2803	0.1647	0.0951	0.0506	0.0208	0.1000
3.2	9545	4598	8110	4444	2535	1480	0851	0452	0185	0890
3.4	9594	4203	7596	4076	2301	1335	0765	0405	0166	0796
3.6	9635	3801	7116	3747	2095	1210	0691	0365	0149	0716
3.8	9671	3396	6670	3452	1914	1100	0627	0331	0135	0648
4.0	1.9701	1.2991	0.6255	0.3187	0.1754	0.1005	0.0571	0.0301	0.0123	0.0588
4.2	9729	2588	5871	2950	1613	0920	0522	0275	0112	0536
4.4	9752	2190	5516	2735	1487	0846	0479	0252	0103	0491
4.6	9772	1798	5187	2542	1375	0780	0441	0232	0094	0451
4.8	9790	1414	4882	2367	1274	0721	0407	0214	0087	0416
5.0	1.9806	1.1038	0.4601	0.2209	0.1184	0.0669	0.0377	0.0198	0.0080	0.0385
5.2	9819	0672	4340	2065	1103	0622	0350	0184	0075	0357
5.4	9832	0315	4098	1934	1029	0579	0326	0171	0069	0332
5.6	9845	0.9970	3875	1815	0963	0541	0304	0159	0064	0309
5.8	9856	9635	3667	1706	0902	0506	0285	0149	0060	0289
6.0	1.9864	0.9310	0.3475	0.1606	0.0847	0.0475	0.0267	0.0139	0.0056	0.0270
6.2	9873	8997	3295	1514	0797	0446	0250	0131	0053	0254
6.4	9880	8694	3129	1430	0751	0420	0235	0123	0050	0238
6.6	9887	8402	2974	1352	0709	0396	0222	0116	0047	0224
6.8	9894	8121	2829	1280	0670	0374	0209	0109	0044	0212
7.0	1.9900	0.7851	0.2694	0.1214	0.0634	0.0354	0.0198	0.0103	0.0042	0.0200
7.2	9905	7591	2568	1153	0601	0335	0187	0098	0040	0189
7.4	9910	7340	2450	1096	0570	0318	0178	0093	0038	0179
7.6	9915	7100	2340	1043	0542	0302	0169	0088	0036	0170
7.8	9919	6869	2237	0993	0516	0287	0160	0084	0034	0162
8.0	1.9923	0.6647	0.2140	0.0947	0.0491	0.0273	0.0152	0.0080	0.0032	0.0154
8.2	9927	6433	2049	0904	0469	0260	0145	0076	0030	0147
8.4	9930	6229	1963	0864	0447	0248	0139	0072	0029	0140
8.6	9932	6032	1882	0827	0427	0237	0133	0069	0028	0133
8.8	9934	5843	1806	0791	0409	0227	0127	0066	0027	0127
9.0	1.9939	0.5661	0.1735	0.0758	0.0391	0.0217	0.0121	0.0063	0.0026	0.0122
9.2	9941	5487	1667	0727	0375	0208	0116	0060	0025	0117
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10.0	1.9950	0.4856	0.1432	0.0620	0.0319	0.0177	0.0099	0.0051	0.0021	0.0099

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